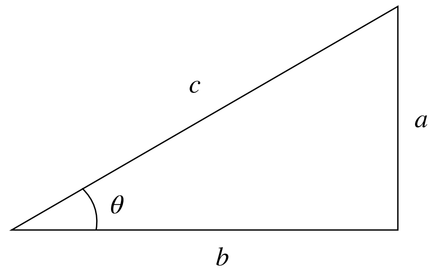
AP Physics C
Calculus

Name____

Trigonometric Functions

1. Consider the right triangle to the right. In terms of *a*, *b*, and *c*, write the expressions for the following:

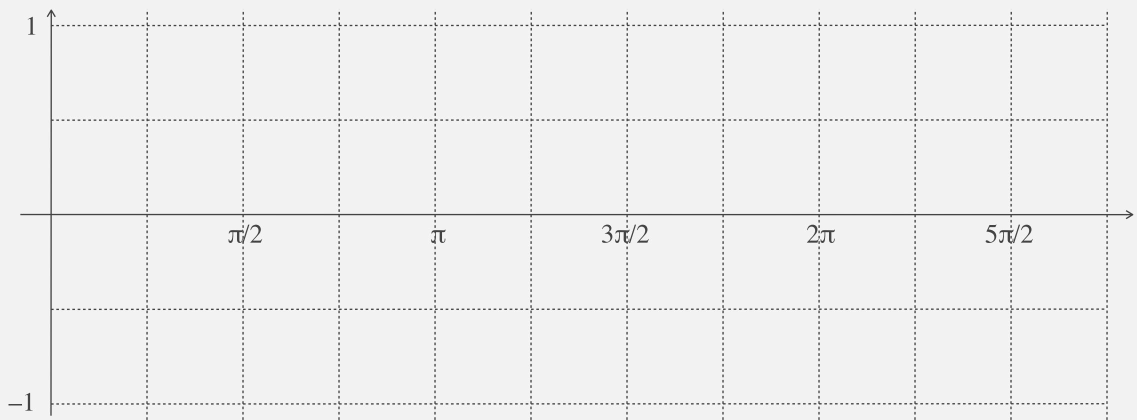


$\sin\theta =$ $\qquad$ $\cos\theta =$ $\qquad$ $\tan\theta =$

2. Use the expressions above and the Pythagorean Theorem to verify that the following identities are true:

$\sin(90° - \theta) = \cos\theta$ $\qquad$ $\cos(90° - \theta) = \sin\theta$ $\qquad$ $\sin\theta\,/\cos\theta = \tan\theta$ $\qquad$ $\sin^2\theta + \cos^2\theta = 1$

3. On the grid below, sketch and label graphs of the following functions: $y = \sin x$, $y = \cos x$, and $y = \sin(x - \pi/2)$.



Recall that the expressions for the sine and cosine of the sum of two angles are:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \qquad \cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

4. Use the above expressions to derive the double angle formulas:

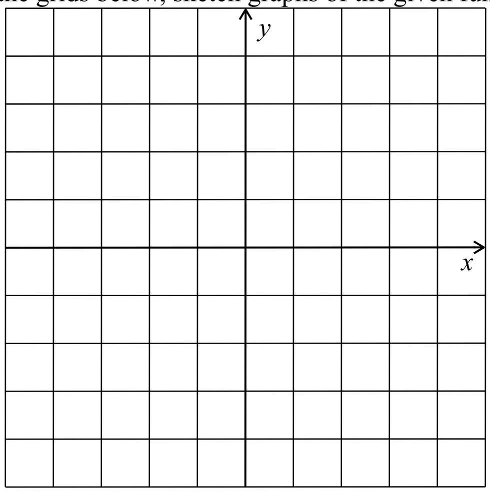
$\sin(2\theta) =$ $\qquad$ $\cos(2\theta) =$

Exponential and Logarithmic Functions

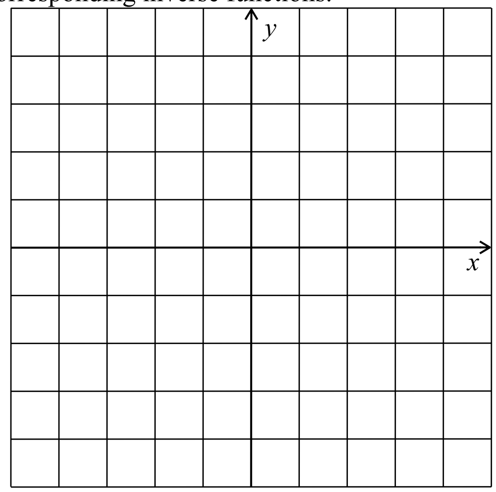
1. Use the rules for logarithms to express the following in terms of log *a* and log *b*, where *a* and *b* are positive numbers:

$\log(ab) =$ $\qquad$ $\log(a/b) =$ $\qquad$ $\log(a^n) =$

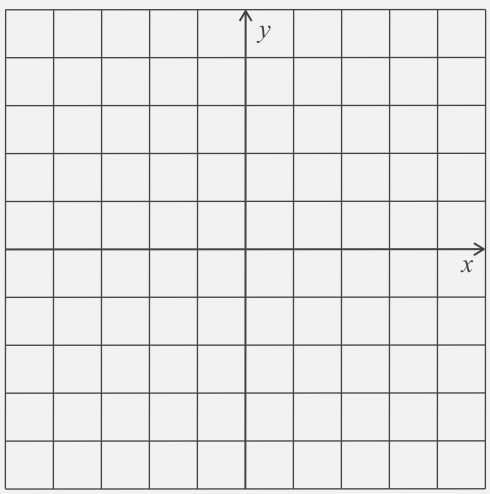
2. On the grids below, sketch graphs of the given functions and the corresponding inverse functions.



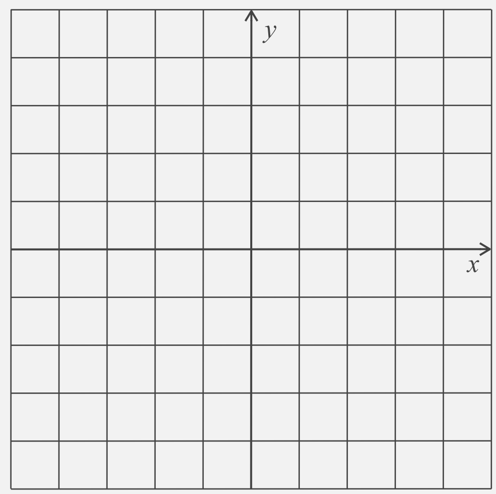
$y = 2x - 4$



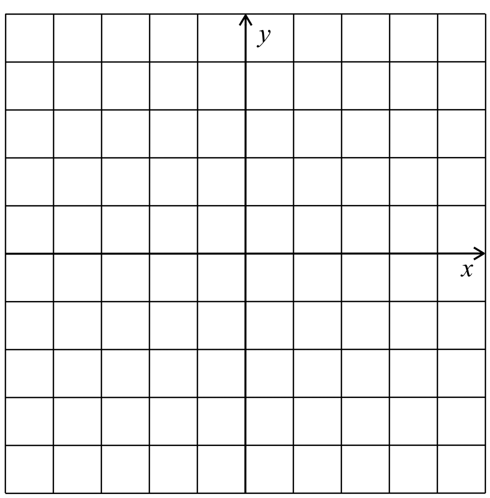
inverse:



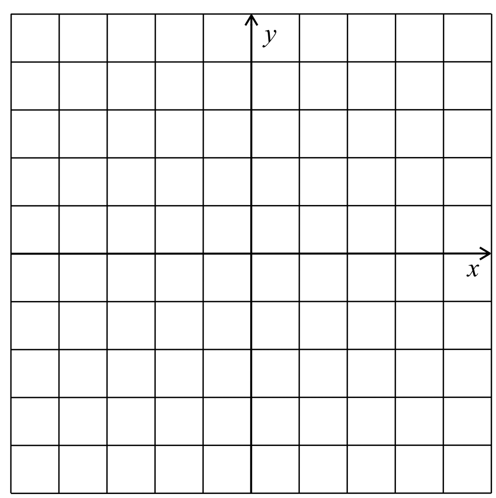
$y = x^2 - 4$



inverse:
(technically not a function)



$y = 2^x$



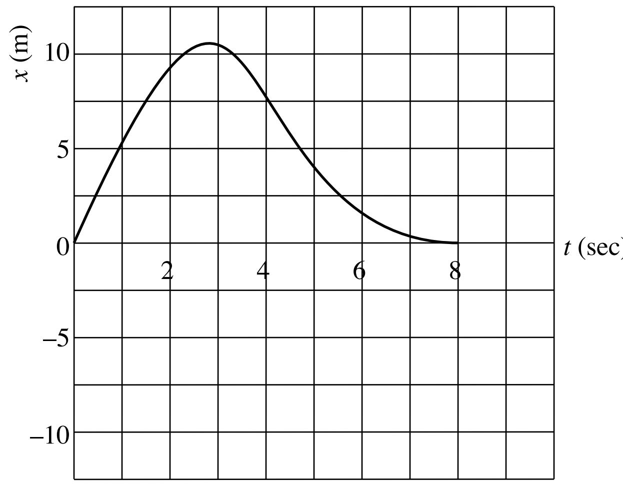
inverse:

Limits

1. Use your calculator to find approximate values of the following limits by entering very small numbers for x:

$$\lim_{x \to 0} \frac{\sin x}{x} = \boxed{} \qquad \lim_{x \to 0} (1 + x)^{1/x} = \boxed{}$$

2. The graph to the right represents the displacement x in meters vs. time in seconds for an object moving along a line. Find the average velocity for the object for the following time intervals:



0 to 8 sec:

1 to 2 sec:

2 to 6 sec:

Describe times or time intervals for which the instantaneous velocity is:

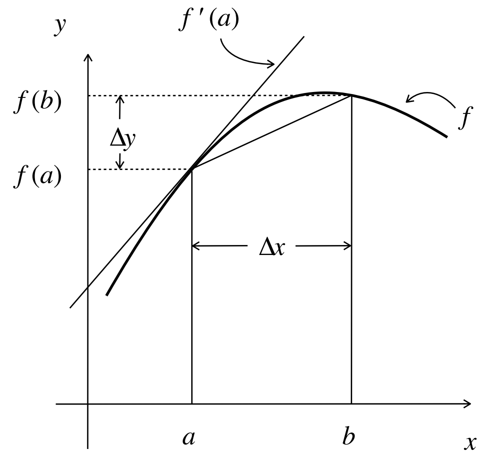
positive:

negative:

zero:

Estimate as best you can the magnitude of the greatest positive velocity and the greatest negative velocity. Explain how you determine them.

Derivatives



Consider a function $y = f(x)$. The *derivative* of f is another function of x; its value at a given point is the slope of the original function at that point. This will be made much more precise in your Calculus class, but we'll attempt a slightly more precise definition here.

Consider two points on the x axis: a and b. The function f has values $f(a)$ and $f(b)$ at these points. The difference between these values is written as Δy, just as the difference between the x values a and b is written as Δx. The slope of the line connecting $(a, f(a))$ with $(b, f(b))$ can then be written different ways. Write the slope in terms of:

1. Δy and Δx:

2. a, b, and f:

3. a, Δx and f:

Imagine that we leave point a where it is and move point b closer to a. We can write this as "$b \to a$".

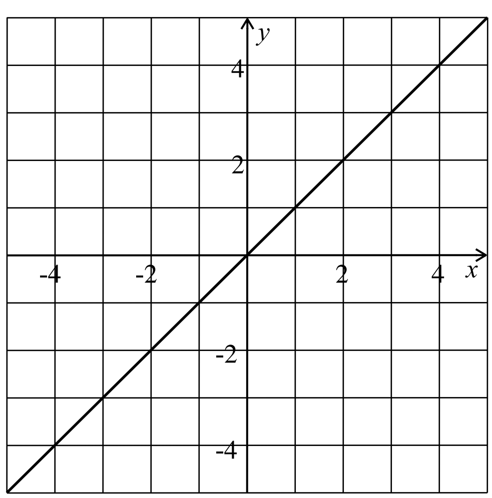
4. How would this be written in terms of Δx?

As point b gets closer to a, the slope gets closer to the slope of the line tangent to f at a (the derivative), which we write as $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$ or $\dfrac{dy}{dx}$ for short. $\dfrac{dy}{dx}$ is read "*dy* by *dx*" or "the derivative of y with respect to x." dy and dx can be thought of as infinitely small versions of Δy and Δx; they are called *differentials*. The derivative of the function f is also written as f'.

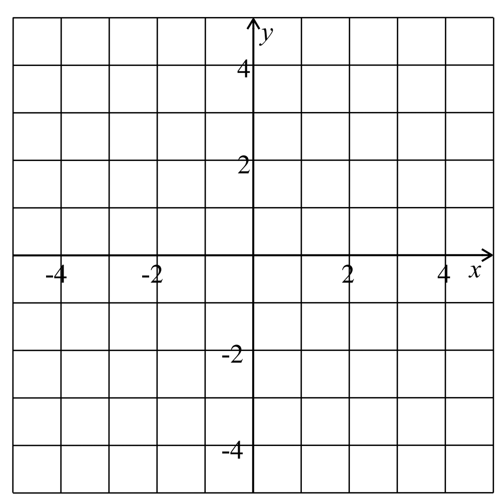
Examples

For each of the following graphs, sketch a graph of the derivative (i.e., the graph of the functions whose values are the slope of the original function at each point).

1.

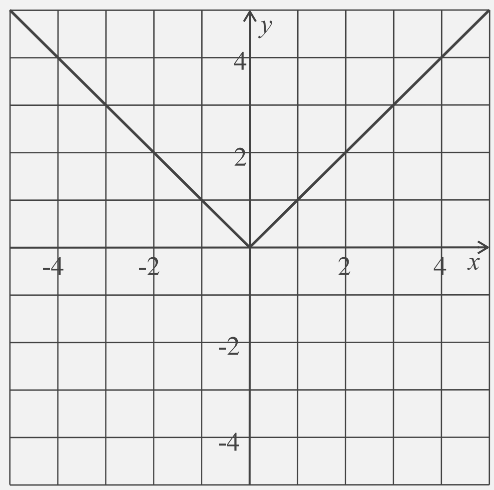


function

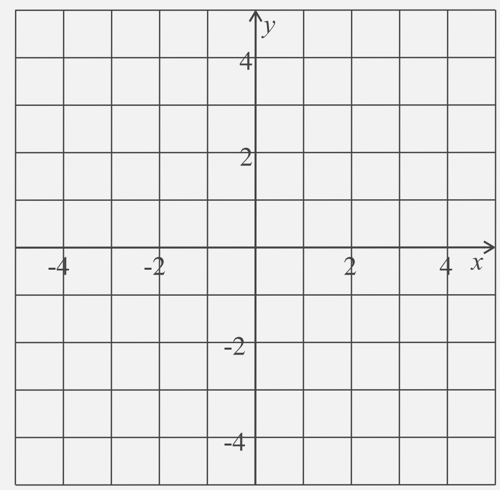


derivative

2.

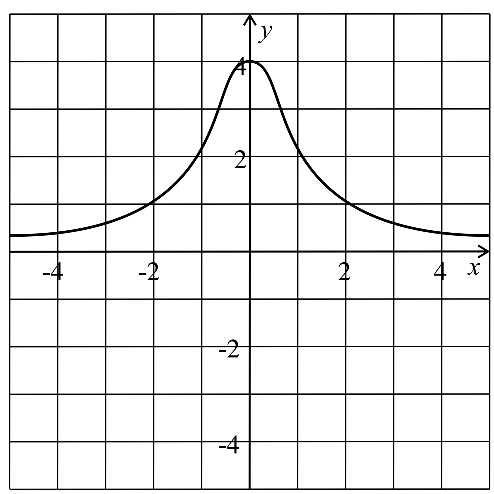


function

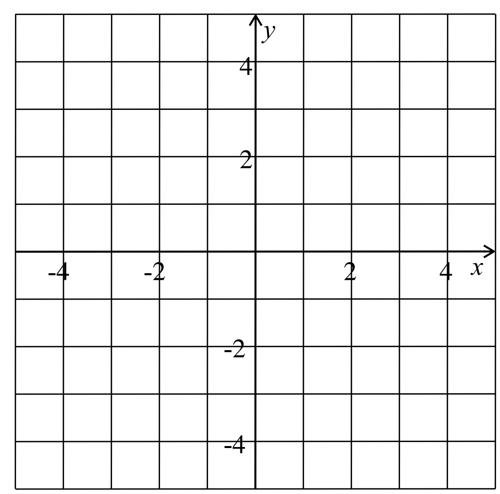


derivative

3.

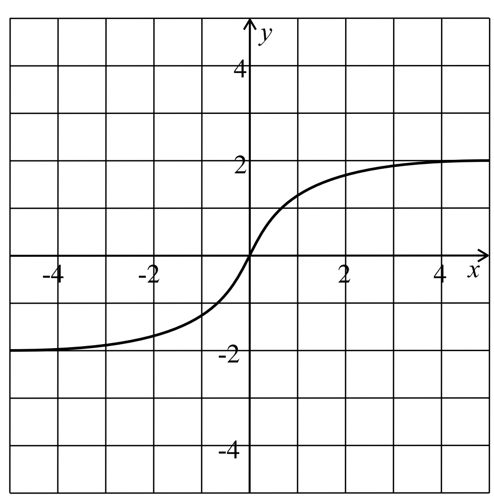


function

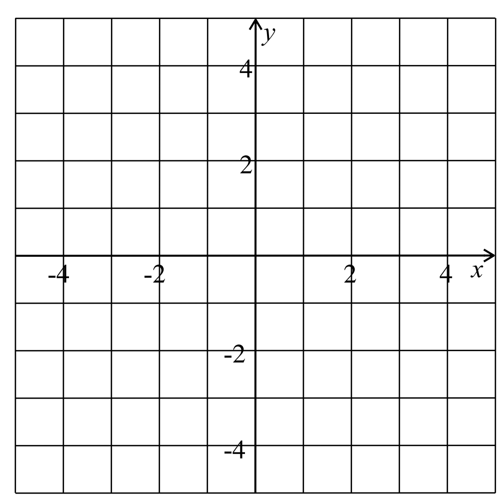


derivative

4.

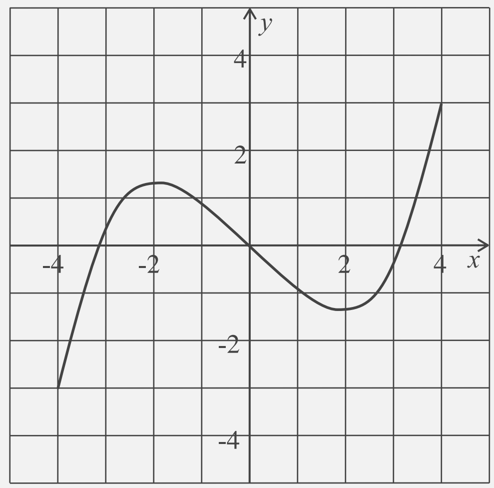


function

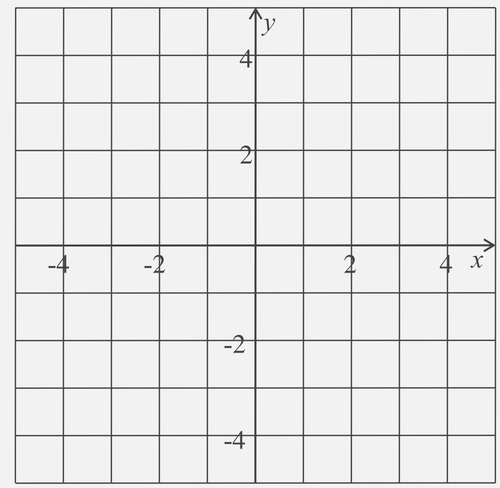


derivative

5.



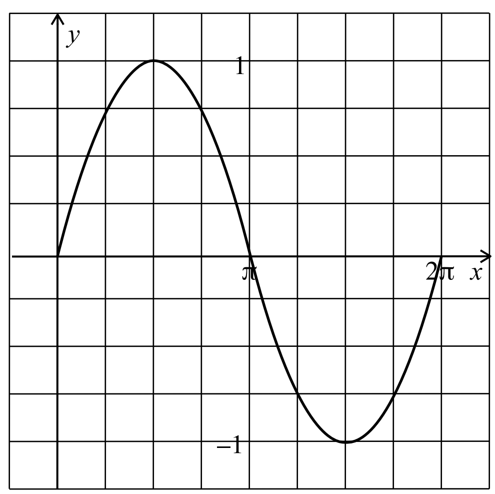
function



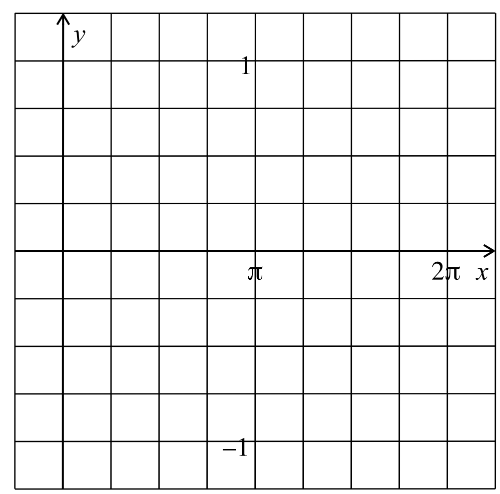
derivative

For the following, guess the derivative function after sketching its graph. Note that for the next two, the vertical and horizontal axes are not scaled the same.

6.

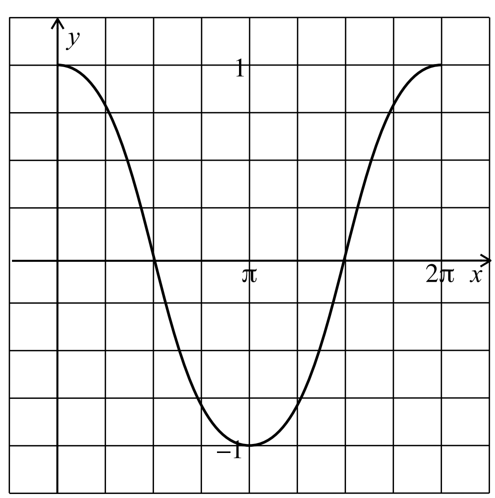


function: $y = \sin x$

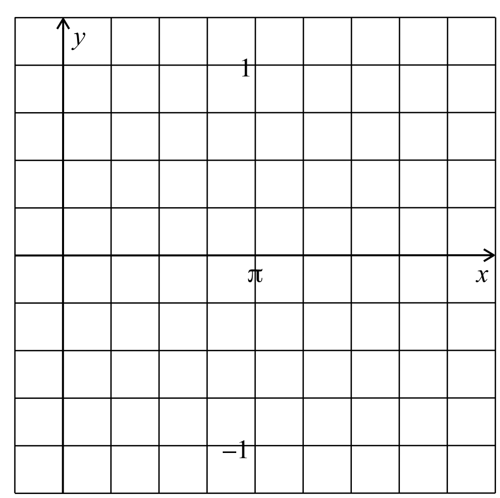


derivative:

7.

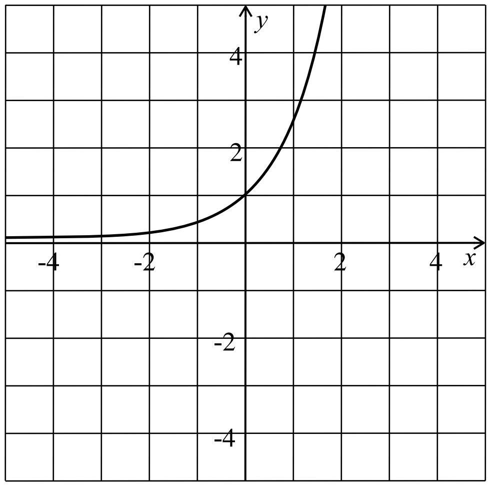


function: $y = \cos x$

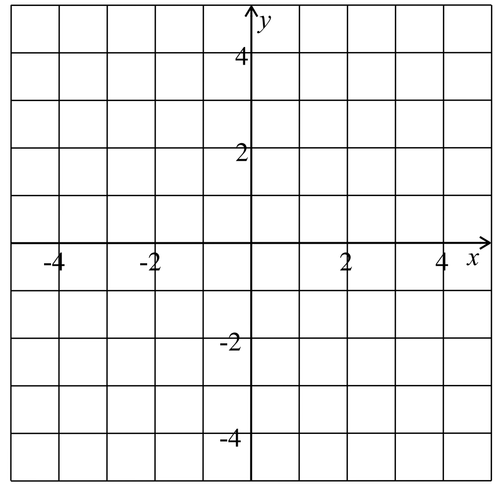


derivative:

8.

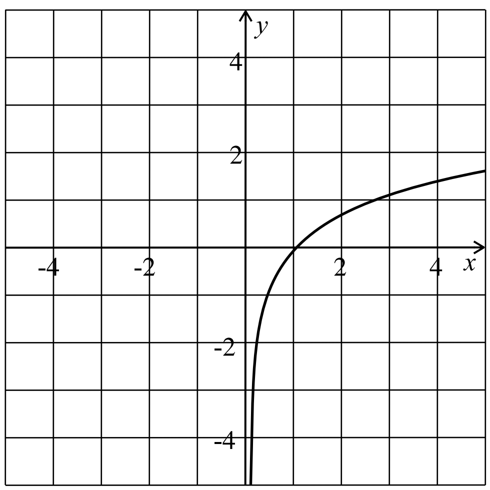


function: $y = e^x$

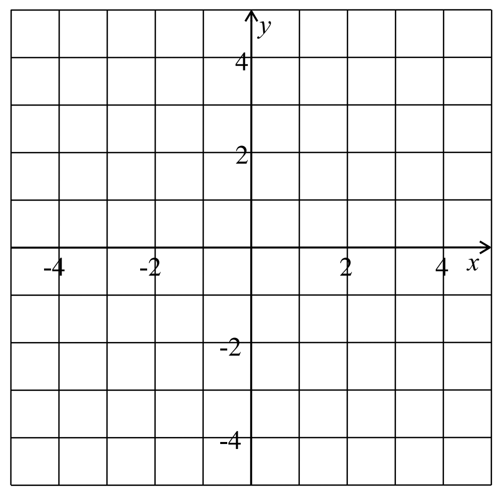


derivative:

9.



function: $y = \ln x$



derivative:

Differentiation

The process of finding the derivative of a function is called *differentiating* the function (not "deriving" the function). The functions you need to be able to differentiate are the power functions (polynomials), trigonometric functions (sin, cos and tan), and e^x and $\ln x$, and functions built from these. In the rules for differentiation to the right, u and v stand for functions of x, and a and m are constants.

Although it's not in the list to the right, we might consider rule zero:

$$0.\ \frac{da}{dx} = 0$$

That is, the derivative of a constant function is zero. Explain why this is so:

1. $\dfrac{dx}{dx} = 1$
2. $\dfrac{d}{dx}(a\,u) = a\dfrac{du}{dx}$
3. $\dfrac{d}{dx}(u + v) = \dfrac{du}{dx} + \dfrac{dv}{dx}$
4. $\dfrac{d}{dx}x^m = m\,x^{m-1}$
5. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$
6. $\dfrac{d}{dx}(u\,v) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
7. $\dfrac{d}{dx}e^x = e^x$
8. $\dfrac{d}{dx}\sin x = \cos x$
9. $\dfrac{d}{dx}\cos x = -\sin x$
10. $\dfrac{d}{dx}\tan x = \sec^2 x$

In addition to the list on the previous page, you need to know the following rules for breaking down complex functions:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$ (called the *quotient rule*: "Lo d-hi minus hi d-lo over lo-lo")

$$\frac{d}{dx}\left(u\left(v(x)\right)\right) = \frac{du}{dv}\frac{dv}{dx}$$ (called the *chain rule*)

The chain rule will be used very frequently to differentiate functions like $y = \sqrt{1+x^2}$, where $v = 1 + x^2$ and $u = \sqrt{v}$.

Examples

Refer to the table of derivatives to differentiate the following functions. Additional instruction is below some problems.

1. $y = x^4 + 8x^3$ $\quad y' = \frac{dy}{dx} =$

2. $y = x^2x^3$ $\quad y' =$ A) B)

(Do this two ways: A) multiply first and B) use the quotient rule)

3. $y = \left(x^5 + 7\right)\left(3x^2 + 1\right)$ $\quad y' =$ A) B)

(Do this two ways: A) eliminate parentheses and B) use the product rule)

4. $y = \frac{1+x}{x^2}$ $\quad y' =$ A) B)

(Do this two ways: A) divide first and B) use the quotient rule)

5. $y = \sqrt{1+x^2}$ $\quad y' =$

(Use the chain rule)

6. $y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$ $\quad y' =$

7. $y = \cos(2x)$ $\quad y' =$

8. What angle does the function above make with the x-axis?

9. $y = \sin(3x) + \cos\left(2x^2\right)$ $\quad y' =$

10. $y = \sin^2(3x - 2)$ $\quad y' =$

For the following, find the higher order derivatives:

11. $y = 2x^3$ $\quad y'' = \dfrac{d^2y}{dx^2} =$

12. $y = x + \dfrac{1}{x}$ $\quad y'' =$

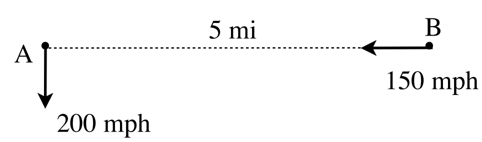
13. $y = x^4$ $\quad \dfrac{d^4y}{dx^4} =$

14. $y = x^2e^x$ $\quad y'' =$

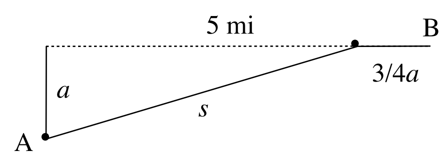
Minima and Maxima

Often a function reaches a local maximum (like the function *f* on page C.3 just before *b*) or a local minimum. The slope of the function (its derivative) would be zero there. We use this to solve problems like the following:

At a certain instant, two airplanes A and B are 5 mi apart. A is due west of B going at 200 mph due south, and B is going 150 mph due west. How close will these planes get to each other? In navigation, this distance is known as the CPA, the Closest Point of Approach.



To solve the problem, express the distance between the planes as a function of some other variable, differentiate the function, and find when the derivative is zero. For example, let plane A cover a little distance a. In that time, B will cover 3/4 that distance because it's going 3/4 as fast.



First, use the Pythagorean Theorem to express the distance s between the planes as a function of a.

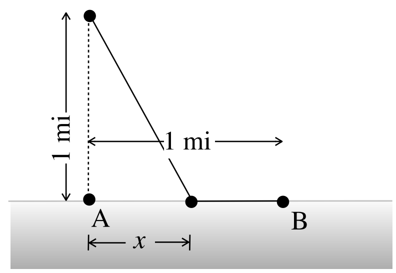
Now find $\dfrac{ds}{da}$, the derivative of s with respect to a.

Set the derivative equal to zero and solve for a; this will tell you how far plane A has gone by the time s is minimized.

Then use this value to find how close they are.

Examples

1. A kayaker finds herself one mile out from a long, straight beach. She wishes to get to a point B on the beach that's one mile along the beach from the point closest to her (point A) as fast as possible. She can kayak at 4 mph, and jog along the beach at 8 mph. Her plan is to paddle to a point a certain distance x from A, leave the kayak and jog the rest of the way.

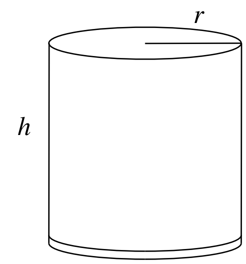


A) What value of x will minimize the time it takes?
(You must express the time t in terms of x, take the derivative of t with respect to x (dt/dx), set it equal to zero and solve for x.)

B) How long will it take her?

2. A glassblower wishes to blow a thin, cylindrical tumbler. The bottom of the tumbler is to be three times as thick as the sides. If the tumbler is to hold a volume V, what are the dimensions (radius and height) that will minimize the amount of glass it takes?

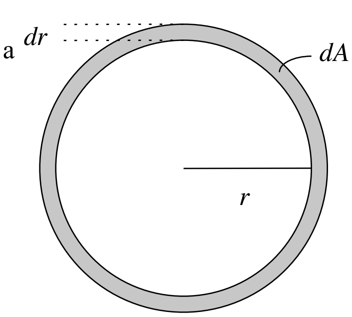
   (Think of the sides of the tumbler as a rectangle, and the bottom as three circles. Express the "area" A of glass as a function of r, and find the value of r that minimizes A. Then find the corresponding value of h. Both r and h should be expressed in terms of V).



Differentials

dx is a *differential*, or an infinitesimal (very small) change in x. For example, if $y = 3x^4$, then $\dfrac{dy}{dx} = 12x^3$. This is called a *differential equation,* an equation involving a function and its derivatives. If the highest derivative that appears is the first derivative, we can treat the dy and the dx as separate things, and solve for dy. This *separates* the differential equation so that it becomes $dy = 12x^3dx$. Another way to say this is that if $y = 3x^4$, then when x changes by an infinitesimally small amount (dx) then y changes by $12x^3$ times as much.

The advantage of differentials is that when things get very small they get simple. For example, consider a circle of radius r as shown to the right. Imagine increasing its radius by a differentially small amount dr. By how much has the area A of the circle changed?



Since dr is so small, this change in area dA can be thought of as a rectangle.

What are the dimensions of this “rectangle”? Length: Width:

Write the differential equation for the area dA:

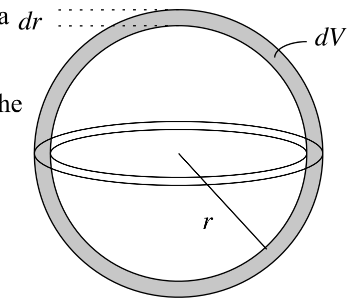
The “differential equation” description of this increase in area is equivalent to another “derivative” version.

What is the area A of the circle? $A =$

What is the derivative of A with respect to r?

Solve this derivative equation for dA to get a differential equation:

Now fill in the blanks to make the analogous statements for a sphere of radius r, considering a differential increase in the volume:



Imagine increasing its radius by a small amount dr. By how much has the volume V of the sphere changed?

Since dr is so small, this change in volume dV can be thought of as a solid shape.

What are the dimensions of this "solid shape"? Area: Height:

Write the differential equation for the volume dV:

The "differential equation" description of this increase in volume is equivalent to another "derivative" version.

What is the volume V of the sphere? $V =$

What is the derivative of V with respect to r?

Solve this derivative equation for dV to get a differential equation:

Integrals

Consider a function $f(x)$. We define the *antiderivative* of f as another function F whose derivative is f; i.e., $\frac{dF}{dx} = f$. For example, if $f(x) = x^n$, then except when $n = 1$, $F(x) = \frac{x^{n+1}}{n+1}$. Explicitly show why this follows from rule #4 on page C.7:

Since the derivative of a constant is zero, specifying f doesn't uniquely determine F, because you can freely add constants to F and its derivative is still f. We use the term *integral* instead of antiderivative, and use the following notation:

$$\int f(x)dx = F(x) + c$$

to mean "the integral of f of x with respect to x is F of x." c is an arbitrary constant called the *constant of integration*. (Often the constant is understood, not explicitly stated.) For example, the expression:

$$\int (\ln x)\, dx = x \ln x - x + c$$

means: The function you take the derivative of to get the function "$\ln x$" is the function "$x \ln x - x$". Use the rules for differentiation to show this:

Refer to the list of integrals to the right. *You should know these integrals by heart. Take the time now to study this list of integrals!*

Note that when you take the derivative of the function on the right, you should get the function that appears between the integral sign and the differential.

For #1, taking the derivative of x with respect to x gives 1, and there is an understood 1 on the left, so that it could read $\int (1)\,dx = x$.

#3 means that polynomials can be integrated term by term.

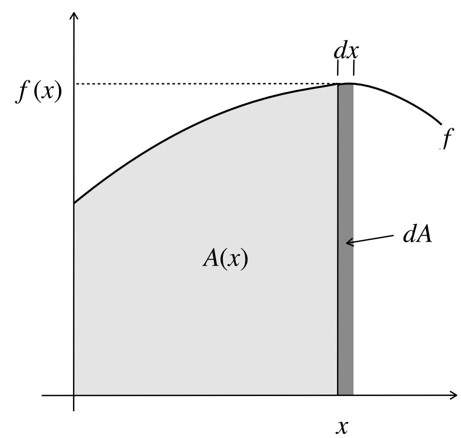
#5 takes care of the exception to #4.

#6 is a special case of #9. #9 will be used *very* frequently in this course.

1. $\int dx = x$
2. $\int a\,u\,dx = a\int u\,dx$
3. $\int (u+v)\,dx = \int u\,dx + \int v\,dx$
4. $\int x^m dx = \dfrac{x^{m+1}}{m+1},\ (m \neq -1)$
5. $\int \dfrac{dx}{x} = \int \dfrac{1}{x}dx = \ln|x|$
6. $\int e^x dx = e^x$
7. $\int \sin x\,dx = -\cos x$
8. $\int \cos x\,dx = -\sin x$
9. $\int e^{-ax}dx = -\dfrac{1}{a}e^{-ax}$

Integrals and Area

Consider an arbitrary function $f(x)$. Define a new function of x, we'll call the *area* function A, as follows: For a given x, find the area under the $f(x)$ curve from the y-axis to a vertical line at x. Area below the x-axis is considered negative. We call this area function $A(x)$.



Now consider increasing x by dx, as we did on page C.11. By how much does the area function increase (dA)?

$dA =$

So then, using integral notation, write an expression for the function A:

$A =$

So just as the derivative of f is the slope of its graph, the integral of f is the area under its graph.

Integration by Change of Variable

Note that although the integral of a sum is the sum of the integrals (#3 in the list above), *there is no product rule, quotient rule or chain rule for integration*. There are various techniques for integrating products, quotients and compositions of functions, one of which is called *change of variable* (also known as a "u-substitution). The steps are these:

1. Pick a new variable (such as u) to represent a portion of the integrand.
2. Find the differential of u (du).
3. Re-express the integrand in terms of u and du.

If this process results in an integral you can evaluate, then do so, otherwise you may have to try again from step 1, letting u represent some other portion of the integrand. For example, consider finding $\int \dfrac{x}{x^2+4}dx$: Let $u = x^2 + 4$. Then what's du? (Hint: find $\dfrac{du}{dx}$ and solve for du.)

Now rewrite the original integral in terms of u and du, with no x's or dx's. Use this to find the integral in terms of u, then substitute back again.

Integration by Separation of Variables

Recall that a *differential equation* can sometimes be separated into an equation involving differentials, such as $dx = x^2 dt$. The solution of a differential equation is a function; in this case $x(t)$. To solve such an equation (find the function x):

A) Separate the variables such that all x's occur on the same side as dx, and all t's (if there were any) are on the same side as dt.

B) Take the integral of both sides.

C) Use the table of integrals to simplify both sides. Each side has an integration constant, but they can be combined.

D) What is the function $x(t)$?

E) Show that this function satisfies the differential equation $dx = x^2 dt$.

Examples

Find the following *indefinite* integrals by change of variable. Include an integration constant. We will see later that integration constants are not needed with *definite* integrals. Use a change of variable for each of these examples. (Each solution should start with "let u = ..., then du = ..." and go from there.)

1. $\int \frac{dx}{1+x} =$

2. $\int \sin^2\theta \cos\theta \, d\theta =$

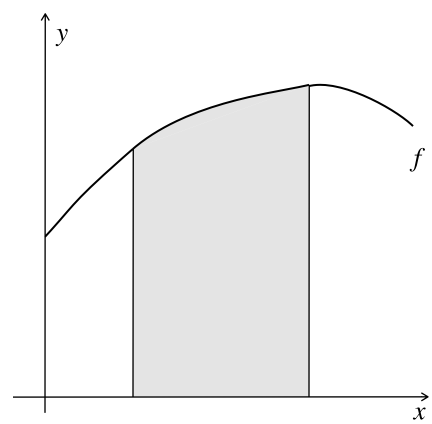
3. $\int x\sqrt{1+x^2}\,dx =$

4. $\int \sin(3x)dx =$

5. $\int \dfrac{8x^2 dx}{\left(x^3+2\right)^3}$

6. $\int x e^{-x^2} dx$

The Definite Integral



Consider the area function A defined above. Now we define a new area function A', as the area under $f(x)$ between two particular values of x; a and b. Clearly this new area function is the original area function at b minus the original area function at a: $A' = A(b) - A(a)$. This new function is called the *definite integral of f between a and b*, and a and b are called the *limits of integration*. If F is the antiderivative of f, then the definite integral is written:

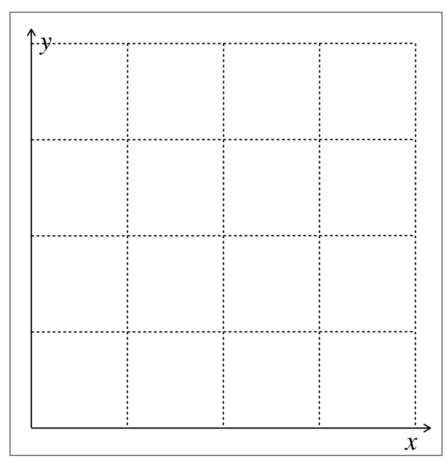
$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)\Big|_a^b$ is just shorthand for evaluating F at the limits of integration, and taking the difference.

Here's an example: Find the area under the curve $y = x^2$ between $x = 1$ and $x = 2$. The solution:

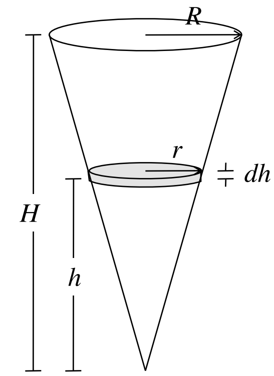
$$\int_1^2 x^2 dx = \frac{x^3}{3}\Bigg|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

Sketch a graph of the function to the right and see whether this result is in the ballpark.



We will always use the definite integral. Explain why we never need to specify the integration constant if we use the definite integral.

Applications of Integration



Integration can be used to find areas and volumes of various shapes. For example, find the volume of a cone of height H and base radius R as follows:

Consider a thin slice of the cone; let the variable h represent the distance of this slice from the vertex. Then the thickness of the slice is a small change in h: dh.

Use similar triangles to find the radius r of the slice in terms of R, H, and h:

The slice has a differentially small volume dV. The thickness of the slice is so small that the sides can be considered vertical, and the radius of the bottom is also r. What is the volume dV of the slice in terms of R, H, and h and the thickness dh?

This is a differential equation. We solve this equation for V by integrating, this time using the definite integral. To find the limits of integration, imagine what range of values h has to have for the slice to sweep over the entire volume we're looking for; these are the two limits. Write the definite integral we need to evaluate:

Evaluate this definite integral to find V.

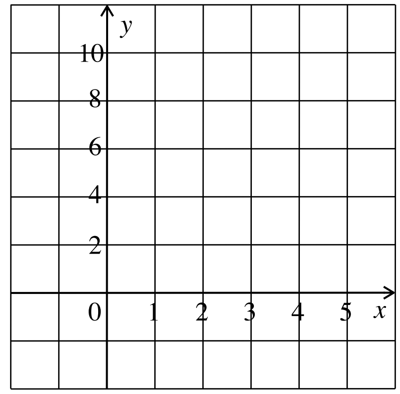
Examples

1. Find the area under the curve $y = \sin x$:
   A) from 0 to π,

   B) from 0 to $\pi/2$,

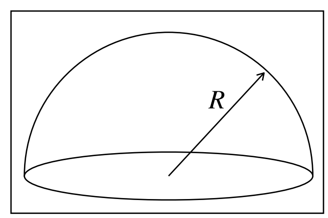
   C) from 0 to 2π.

2. Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.
   A) Make a sketch of the two functions to the right.
   B) *Describe* (in words) how you can use definite integrals to find the area bounded by these two curves. Be specific about the limits of integration.



   C) Translate your description into an expression involving the corresponding definite integrals.
   D) Simplify this expression as much as possible *before* integrating, then perform the integration.

3. Find the volume of a hemisphere of radius R by dividing it into thin disks and integrating their volumes (similar to the cone on page C.16). *You must clearly define, and draw in the diagram, any variables you need.*



Exercises

Finding Derivatives

1. $y = 2x^{1/2} + 6x^{1/3} - 2x^{3/2}$ $\quad \dfrac{dy}{dx} =$

2. $s = \left(t^2 - 3\right)^4$ $\quad \dfrac{ds}{dt} =$

3. $f(t) = (2t - 1)\sqrt{3 - t^2}$ $\quad \dfrac{df}{dt} =$

4. $x = y\sqrt{1 - y^2}$ $\quad \dfrac{dx}{dy} =$

5. $y = \tan x^2$ $\quad \dfrac{dy}{dx} =$

6. $y = \tan^2 x$ $\quad \dfrac{dy}{dx} =$

7. $f(x) = x^2 \sin x$ $\quad \dfrac{df}{dx} =$

8. $y = \ln\left(3x^2 - 5\right)$ $\quad \dfrac{dy}{dx} =$

9. $y = \ln(x+3)^2$ $\quad \dfrac{dy}{dx} =$

10. $y = \ln^2(x+3)$ $\quad \dfrac{dy}{dx} =$

11. $y = \ln\big(\sin(3x)\big)$ $\quad \dfrac{dy}{dx} =$

12. $f = \ln\big(\sin\theta\big)$ $\quad \dfrac{df}{d\theta} =$

13. $y = x\ln(x) - x$ $\quad \dfrac{dy}{dx} =$

14. $y = \sqrt{1+x^2}$ $\quad \dfrac{dy}{dx} =$ $\quad \dfrac{d^2y}{dx^2} =$

15. $f = \tan\theta$ $\quad \dfrac{df}{d\theta} =$ $\quad \dfrac{d^2f}{d\theta^2} =$

16. $y = xe^{x^2}$ $\quad \dfrac{dy}{dx} =$ $\quad \dfrac{d^2y}{dx^2} =$

17. Find the slope of the curve $x = y^2 - 4y$ at the points where it crosses the y-axis.

18. Find the equation for the line tangent to the curve $y = \dfrac{\ln x}{x}$ at $x = \sqrt{e}$.

19. A point moves along the curve $y = x^3 - 3x + 5$ so that $x = \dfrac{\sqrt{t+3}}{2}$, where t is time. At what rate is y changing when t = 6 (in arbitrary units)?

20. Apple Inc. is coming out with a new iPhone model, and they wish to maximize the gross income from its sales. If N iPhones are sold at a price p, the gross income is Np. However, as the price is raised, the number of buyers decreases. It is estimated that the number of buyers at price p can be described by the expression to the right, where N_o and p_o are constants. Note that as the price approaches p_o the sales approach zero; the expression is meaningless for $p > p_o$. What should be the price for the maximum income, and what is the maximum income?

$$N(p) = N_o\left[1 - \left(\frac{p}{p_o}\right)^2\right]$$

Indefinite Integrals

21. $\int \frac{dx}{x^{2/3}} =$

22. $\int \left(2x^2 - 5x + 3\right) dx =$

23. $\int (1 - x)\sqrt{x}\, dx =$

24. $\int \frac{x^2}{\sqrt[4]{x^3 + 2}} dx =$

25. $\int \frac{\ln x}{x} dx =$

26. $\int \sin x\, e^{\cos x} dx =$

Definite Integrals

27. $\int_0^3 x^2 dx =$

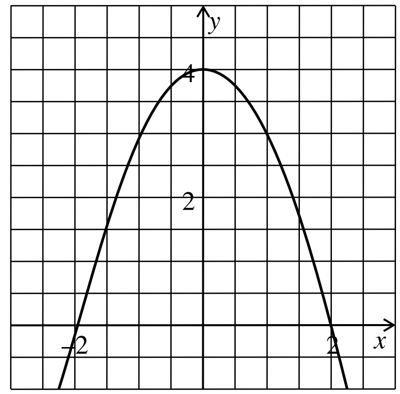
28. $\int_{-1}^{1} \left(2x^2 - x^3\right) dx =$

29. $\int_1^2 (2 - 3x)^3\, dx =$

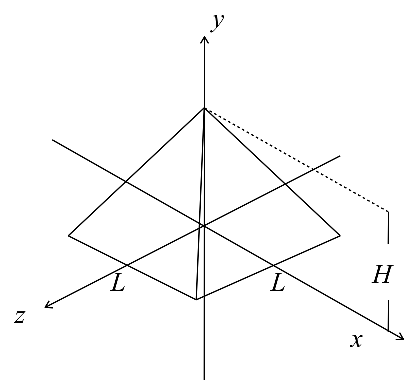
30. $\displaystyle\int_0^4 \frac{1}{x+5}\,dx =$

31. $\displaystyle\int_{-1}^1 \frac{e^x}{1+e^x}\,dx =$

32. Find the area bounded by the curve $y = -x^2 + 4$ and the x-axis.



33. Find the volume of a pyramid with a square base of side L and a perpendicular height of H.



AP Physics C
Calculus Practice Quiz

Name ________________________________

Part I
Short Answer (1 point each)

1. Given $y = 7x^4$, find d^2y/dx^2 1. ____________________

2. Given $y = x^2 e^x$, find dy (not dy/dx). 2. ____________________

3. Given $y = \left(x^3 + 5\right)^2 x^4$, find dy/dx 3. ____________________
(you need not simplify)

4. Given $y = \sin\left(\sqrt{1 + x^2}\right)$, find dy/dx 4. ____________________

Evaluate the following indefinite integrals.

5. $\int \frac{1}{x^5}\,dx$ 5. ____________________________

6. $\int x^2\left(1-x^3\right)^2\,dx$ 6. ____________________________

7. $\int \frac{x\,dx}{x^2-1}$ 7. ____________________________

Evaluate the following definite integrals. *Express as decimal numbers to the nearest hundredth*.

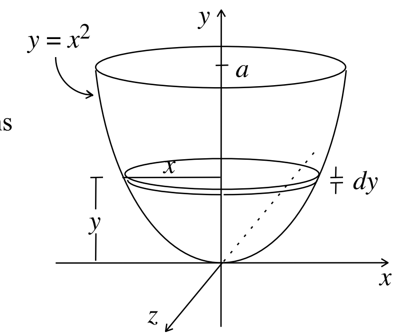
8. $\int_{-1}^{1}\left(2x^2-x^3\right)dx$ 8. ____________________________

9. $\int_{0}^{4}\frac{1}{x+6}\,dx$ 9. ____________________________

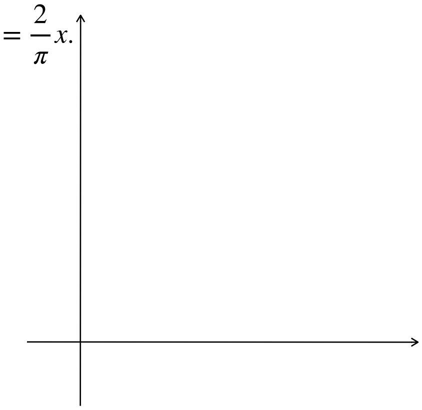
10. $\int_{-1}^{1} e^{-\frac{x}{2}}\,dx$ 10. ____________________________

Part II
Show your work (5 points each)
Credit depends on the quality and clarity of your explanations

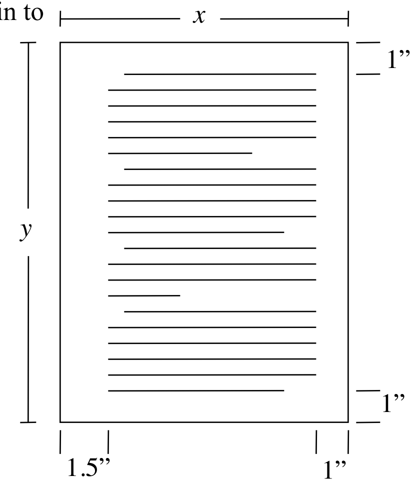
1. Consider the shape formed by rotating the parabola $y = x^2$ about the y-axis. Find the volume bounded by that curve and the plane $y = a$. Consider a disc of radius x and thickness dy at a distance y from the x-axis (drawn to the right). Write a differential equation involving dV, the differential volume of the disc, and integrate with appropriate limits to find the volume. Express your solution in terms of a and π.



2. Find the area in the first quadrant bounded by the curve $y = \sin x$ and the line $y = \dfrac{2}{\pi}x$. Draw a rough sketch of the graph on the axes to the right.



3. A printer is to use a page with a total area of 80 in^2. The left margin is to be 1.5 in to allow for binding , but the top, right and bottom margins are all to be 1 in. Find the dimensions of the page (x and y) such that the area of the actual printed matter is a maximum.



AP Physics C
Unit 1 Name________________________

Measurement

In physics we study what the world is made of and how these components interact. Physicists make observations, find patterns, deduce general relationships and make predictions based on a commonly agreed upon system of measurement called the SI (Système International d'unités) system. The fundamental units in the SI system are the *meter* (a unit of length), the *kilogram* (a unit of mass) and the *second* (a unit of time). Using only these three units (and one or two others), we can derive units for any quantity we care to measure.

The choice of length, mass and time as fundamental quantities is somewhat arbitrary, based on how convenient it is to define units. We can define other quantities as fundamental and derive these. For example, imagine another system of units, the Planck system, in which the fundamental units are the "G" ($6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg} \cdot \mathrm{s}^2}$), the "$c$" ($3.00 \times 10^8$ m/s) and the "h" ($6.63 \times 10^{-34} \frac{\mathrm{kg} \cdot \mathrm{m}^2}{\mathrm{s}}$). In this system of units, find the following units and their SI equivalents:

A) The derived unit of length (called the Planck Length):

B) The derived unit of mass (called the Planck Mass):

C) The derived unit of time (called the Planck Time):

D) The derived unit of mass density:

A typical man's beard grows at the rate of 1/32 inch per day. What is the growth rate R, expressed in Planck units? (Show the unit conversion; 1 inch is 2.54 cm.)

Mechanics

Mechanics is the study of motion, and we begin by distinguishing between a pure description of the motion (kinematics) and an explanation of the causes of motion (dynamics). To simplify our study we first consider kinematics.

The description of the motion of an object can be very complex, but can be described in three main ways. For example, consider an diver in an olympic diving competition. During the course of the dive, she changes shape (tucks and untucks), translates (her whole body moves from one place to another) and rotates or spins. We wish to consider the simplest kind of motion first, so we make the following restrictions:

1. We consider only rigid objects; ie., each point on the object retains its position relative to every other point.
2. We consider non-rotating objects — we take up the study of rotation later.
3. We consider translation in one dimension (positive and negative) only.

Considering the object as a *particle* allows us to obey restrictions 1 and 2, and concentrate on the motion of the object as a whole. Each point on a rigid, non-rotating object travels with the same trajectory (path) as every other point.

One-Dimensional Particle Kinematics

For a complete description of the motion of a particle in one dimension, we need only define two fundamental quantities, and use the power of calculus that we have just learned. These quantities are:

1. *Position* - We superimpose a "number line" or axis along the direction of motion of the particle, choosing an arbitrary origin and +/– directions. The position of the particle is its coordinate on this axis, symbolized by x.
2. *Time* - We imagine a stopwatch which we can start at an arbitrary time and measure time elapsed. The coordinate of the particle on the time axis we call t.

We now use calculus to define two derived quantities. Using the symbols defined above, and your knowledge of calculus, write expressions for these quantities:

3. *Velocity* - The rate of change of position with time:
4. *Acceleration* - The rate of change of velocity with time: , which can also be written in terms of the position as:

Acceleration is the key concept in kinematics. We start by writing the defining equation for acceleration (the first equation in #4 above) as a *differential equation*: The goal is to write equations that tell where the particle is (x) and how fast (and in what direction) it's moving (v) at any time (t), given only the *initial conditions* and this differential equation, called the *equation of motion*. The initial conditions are its initial position x_o, (spoken "x-naught"), and velocity v_o, (spoken "v-naught"). Since we are free to start measuring time intervals at any moment, we usually choose the initial time to be $t = 0$ *without any loss of generality.*

Take the definite integral of both sides of the equation of motion, choosing appropriate limits of integration. The lower limits should represent the initial conditions, and the upper limits should represent conditions at time t:

Can either or both of these integrals be evaluated? If so, evaluate; if not, explain why not.

Kinematics with Constant Acceleration

Assuming that acceleration is constant, evaluate the integrals on both sides of the equation above, and solve for v as a function of time.

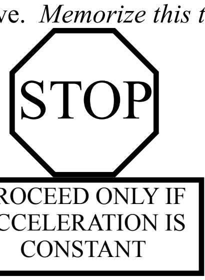
Write the equation you just derived as a differential equation in x, with variables properly separated.

Choose appropriate limits of integration and integrate once again to obtain an equation for x as a function of t.

Combine the equations above for $v(t)$ and $x(t)$ to eliminate a.

Combine the equations above for $v(t)$ and $x(t)$ to eliminate t.

Complete the table to the right by filling in the four kinematic equations you derived above. *Memorize this table!*



It is a common mistake to assume that acceleration is constant when it is not!

Variable left out	Kinematic Equation for Constant Acceleration
x	
v	
a	
t	

Discuss and answer the following questions:

A) Can an object have zero velocity and still be accelerating? If so, give an example.

B) Can an object reverse the direction of its velocity while its acceleration is constant? If so, give an example.

C) Can an object be increasing its speed while its acceleration is decreasing? If so, give an example.

D) In time trials for the Indianapolis 500 mile race, a car completes the first lap with an average speed of 100 mph. The driver wishes to complete the second lap so that the average speed for the two laps is 200 mph. Can this be done? If so, how fast must he drive the car for the second lap?

Examples

Refer to the table on the previous page in solving the next problems, which involve constant acceleration. In each case, make a table (or tables) showing the variables x, a, t, v_o, and v. Fill in the values known from the problem, and put a "?" by the variable whose value you're looking for. Choose the equation(s) to write down based on which of the 5 variables is left out. Solve the equation(s) and plug in.

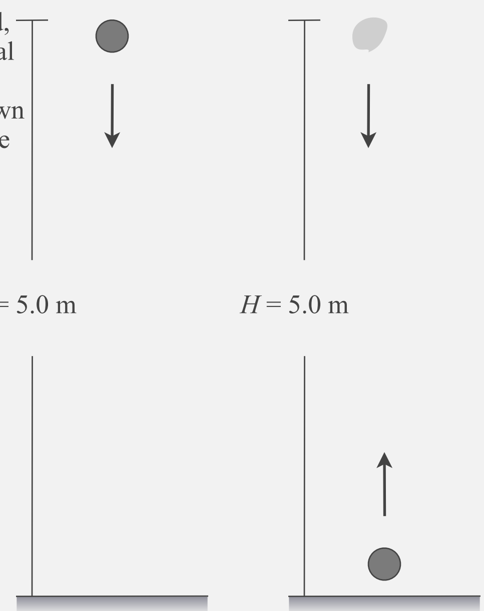
1. An elementary particle is traveling with an initial velocity of 5.2×10^6 m/s into a region where an electric field causes an acceleration of 1.3×10^{14} m/s^2 directed opposite to the velocity. How far does the particle go before coming to rest?

2. In a speed trap, two pressure-activated strips are placed 110 m apart across a highway on which the speed limit is 90 km/hr. While going 120 km/h, a driver notices the police car just as he activates the first strip. What (negative) acceleration is needed so that the system measures a legal speed?

3. At the instant a stoplight turns green, a car starts accelerating at 2.2 m/s^2. At the same instant in the next lane, a truck going a constant 9.5 m/s passes the car.
   A) How far from the car's starting point will it pass the truck? (This problem involves making two tables and solving two kinematic equations simultaneously.)

   B) How fast will the car be going at that instant?

4. A superball is dropped from rest at a height $H = 5.0$ meters above the ground, as shown to the right. When it hits the ground, it rebounds with a speed equal to that when it hit. At the instant the ball rebounds, a small blob of clay is released from rest from the original height H, directly above the ball, as shown on the right. The clay blob, which is descending, eventually collides with the ball, which is ascending. Recall that objects in free fall accelerate at $g = 10\ \mathrm{m/s^2}$, and neglect air resistance.



   A) Determine the speed of the ball immediately before it hits the ground.

B) Determine the time after release of the clay blob at which the collision takes place.

C) Determine the height above the ground at which the collision takes place.

D) Determine the speeds of the ball and the clay blob immediately before the collision.

5. A two-stage rocket leaves its launch pad moving vertically upward with a uniform acceleration of 4 m/s^2. At ten seconds after launch, the first stage of the rocket runs out of fuel and separates from the second stage, whose engine gives it an upward acceleration of 6 m/s^2.

A) How high above the launch pad is the rocket when the first stage separates?

B) How fast is the rocket moving when the first stage separates?

C) What is the maximum altitude of the first stage of the rocket?

D) What is the distance between the first and second stages at 2 s after separation?

E) Answer part D) by considering the *relative* acceleration and *relative* initial velocity of the stages.

Kinematics with Non-Constant Acceleration

Recall (page 1.2) the integral of the equation of motion. If the acceleration is not constant, it cannot come out of the integral. In general, acceleration may vary depending on time or the velocity of the object. In these cases, we must know how the acceleration varies so that we can integrate to find the velocity and position at the general time *t*.

Examples

1. The position of a particle along the x axis depends on time according to the equation $x = At^2 - Bt^3$, where x is in meters and t is in seconds.

   A) What SI units must the constants A and B have?

   B) Derive the equation for the velocity of the particle as a function of time.

   C) Show that this equation is dimensionally correct.

   D) Derive the equation for the acceleration of the particle as a function of time.

   E) Show that this equation is dimensionally correct.

2. A particle starts at $x_o = 0$ and moves along a straight line so that its velocity is given by the equation $v = At + \dfrac{B}{t + C}$, where A, B and C are constants.

   A) find the particle's initial velocity.

   B) Write the general expression for the particle's acceleration as a function of time.

C) Write the general expression for the position as a function of time.

3. A particle starts at the origin at $t = 0$ and moves along a straight line so that its position x is given by the equation $x = At^4 - Bt$, where A, and B are constants.

A) What are the SI units of the constants A and B?

Assume that the constants A and B have the numeric values 6 and 2, respectively.

B) At what time is the particle at its farthest negative distance from the origin?

C) How far is the particle from the origin at this time?

D) What is the acceleration of the particle at this time?

4. A particle's velocity is described by the function $v = kt^2$ where v is in m/s and t is in s. The particle's position at $t = 0$ is –9 m, and its position at $t = 3$ s is +9 m.

A) Determine the value of the constant k, with units.

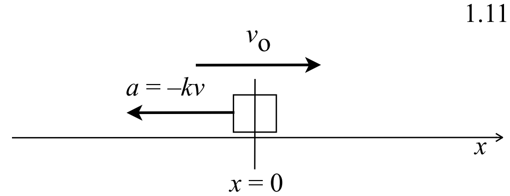
B) Determine the velocity of the particle at $t = 0$ and $t = 3$ s.

C) Determine the acceleration of the particle at $t = 0$ and $t = 3$ s.

D) Develop an expression for the position x of the particle as an explicit function of time t.

E) Find the time at which the particle is at the origin ($x = 0$).

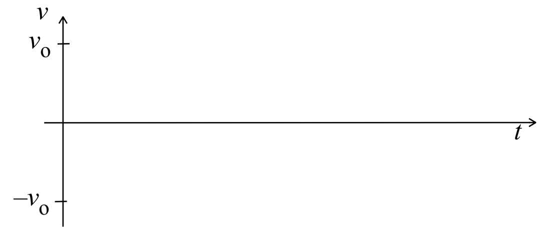
5. An object moving along the x-axis with velocity v is slowed by an acceleration $a = -k\,v$, where k is a positive constant. At time $t = 0$, the object has velocity v_o at position $x = 0$, as shown to the right.



A) What are the correct SI units for the constant k?

B) What is the initial acceleration a_o of the object?

C) Derive an equation for the object's velocity as a function of time t, and sketch this function on the axes to the right. Let a velocity directed to the right be considered positive.



D) Derive an equation for the object's acceleration as a function of time t, and sketch this function on the axes to the right. Let an acceleration directed to the right be considered positive. Label significant points on the vertical axis.

E) Derive an equation for the distance the object travels as a function of time t and sketch this function on the axes to the right. Label significant points on the vertical axis.

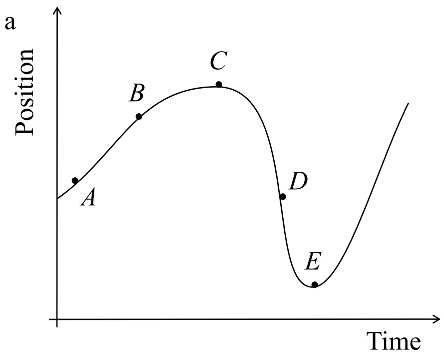
F) Determine the distance the object travels from $t = 0$ to $t = \infty$.

AP Physics C
Unit 1 Practice Test

Name ____________________

Part I
Multiple Choice

For questions 1 - 3: The graph to the right represents the position vs. time graph for a particle moving in a straight line.



1. At which of the labeled points is the magnitude of the velocity greatest?

   A) *A* B) *B* C) *C* D) *D* E) *E*

2. At which of the labeled points is the velocity zero?

   A) *B* only B) *E* only C) *D* only D) *C* and *D* E) *C* and *E*

3. At which of the labeled points is the magnitude of the acceleration greatest?

   A) *A* B) *B* C) *C* D) *D* E) *E*

4. Suppose $a = b^x c^y$, where a is in m·s, b is in m^2/s and c is in m·s^2. Then the exponents x and y have the values:

   A) 2/3 and 1/3
   B) 2 and 3
   C) 4/5 and –1/5
   D) 1/5 and 3/5
   E) 1/2 and 1/2

5. The area under the acceleration vs. time graph for a certain time interval of an object's motion represents

   A) The average velocity of the object during the time interval
   B) The instantaneous velocity of the object at the end of the time interval
   C) The average speed of the object during the time interval
   D) The change in velocity of the object during the time interval
   E) The object's velocity at the time midway through the time interval

6. A ball initially at rest falls without air resistance from a height h above the ground. If the ball falls the first distance $\frac{h}{2}$ in time t, then the time to fall the remaining distance of $\frac{h}{2}$ is

   A) $\frac{t}{4}$

   B) $\left(\sqrt{2} - 1\right) t$

   C) $\frac{t}{2}$

   D) $\frac{\sqrt{2}}{2} t$

   E) t

7. The coordinate of an object moving in one dimension is given as a function of time by $x = 8t - 3t^2$, where x is in meters and t is in seconds. Its average velocity over the interval from $t = 1$ s to $t = 2$ s is

A) –2 m/s
B) –1 m/s
C) –0.5 m/s
D) 0.5 m/s
E) 1 m/s

8 Two objects both move and uniformly accelerate to the right. At time $t = 0$, the objects are at the same initial position, but:

Object 1 has an initial speed twice that of Object 2 and
Object 1 has one-half the acceleration of Object 2.

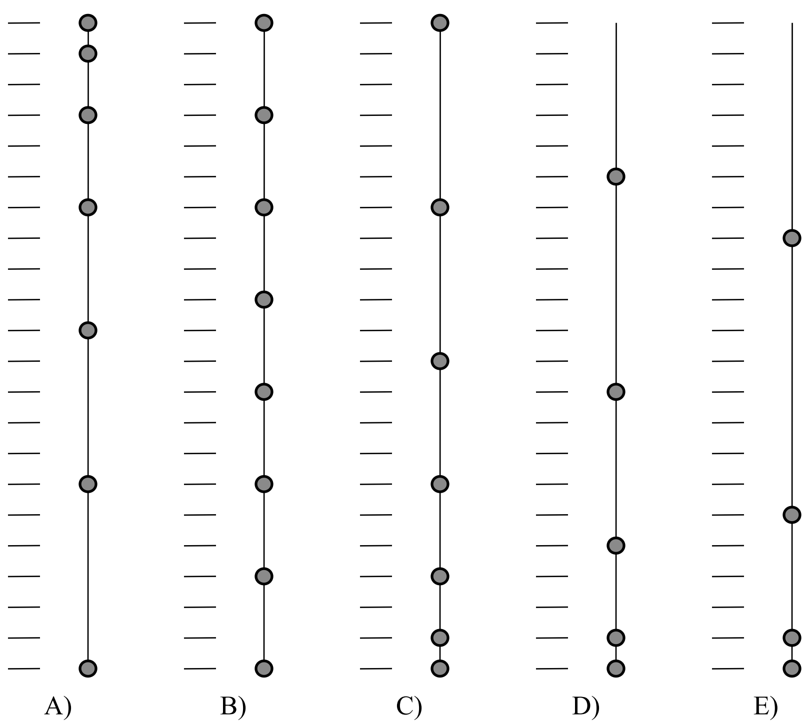
After some time T, the velocity of the two objects is the same. What is the ratio of the distance traveled in this time by Object 2 to that traveled by Object 1?

A) 5:6
B) 4:5
C) 3:4
D) 2:3
E) 1:2

9. A person standing on the edge of a fire escape simultaneously launches two apples, one straight up with a speed of 7 m/s and the other straight down at the same speed. How far apart are the two apples 2 seconds after they were thrown, assuming that neither has hit the ground?

A) 14 m
B) 20 m
C) 28 m
D) 34 m
E) 56 m

10. You have 5 different strings with weights tied at various point, all hanging from the ceiling, and reaching down to the floor. The string is released at the top, allowing the weights to fall. Which one will create a regular, uniform beating sound as the weights hit the floor?



11. Two cars traveling towards each other are 150 km apart. One car is moving at 60 km/hr and the other is moving at 40 km/hr. They will meet in

    A) 2.5 hr
    B) 2 hr
    C) 1.75 hr
    D) 1.5 hr
    E) 1.25 hr

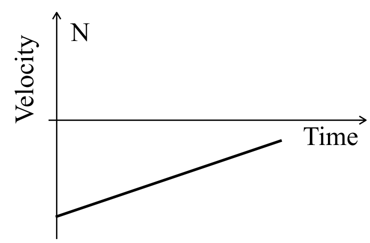
12. Each of four particles moves along the *x*-axis. Their coordinates are given as functions of time below:

    particle 1: $x(t) = 3.5\,\mathrm{m} - 2.7\,\mathrm{m/s^3}\,t^3$ particle 2: $x(t) = 3.5\,\mathrm{m} + 2.7\,\mathrm{m/s^3}\,t^3$
    particle 3: $x(t) = 3.5\,\mathrm{m} - 2.7\,\mathrm{m/s^2}\,t^2$ particle 4: $x(t) = 3.5\,\mathrm{m} - 3.4\,\mathrm{m/s}\,t - 2.7\,\mathrm{m/s^2}\,t^2$

    Which of these particles have a constant acceleration?

    A) all four B) 1 and 2 only C) 2 and 3 only D) 3 and 4 onlyE) none of them

13. The velocity–time graph to the right represents a car on a freeway. North is defined as the positive direction. Which of the following describes the motion of the car?



    A) The car is traveling north and slowing down.
    B) The car is traveling south and slowing down.
    C) The car is traveling north and speeding up.
    D) The car is traveling south and speeding up.
    E) The car is traveling northeast and speeding up.

For questions 14 and 15: A particle moving in one dimension has a position function defined as:

$$x(t) = 6\,\mathrm{m/s^4}\,t^4 - 2\,\mathrm{m/s}\,t$$

14. The particle reaches its maximum displacement at

    A) $t = 0.44$ s
    B) $t = 0.29$ s
    C) $t = 0.083$ s
    D) $t = 0.69$ s
    E) The particle doesn't have a maximum displacement.

15. When the acceleration of the particle is 12 m/s^2, its velocity is

    A) –0.41 m/s
    B) 0.41 m/s
    C) –0.37 m/s
    D) 0.37 m/s
    E) The particle's acceleration is never 12 m/s^2.

1.D 2.E 3.E 4.D 5.D 6.B 7.B 8.B 9.C 10.D 11.D 12.D 13.B 14.E 15.C

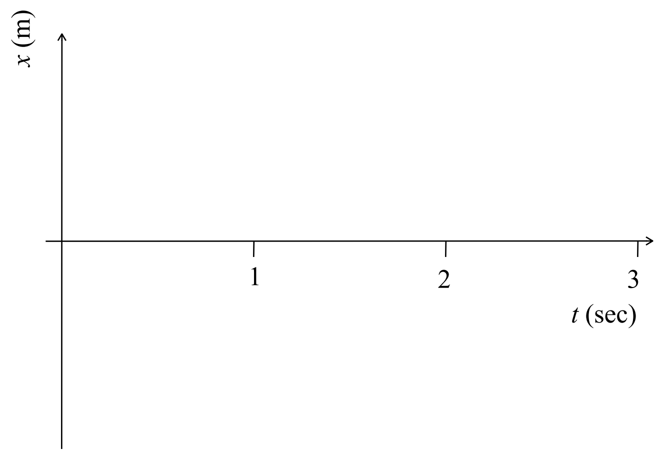
Part II

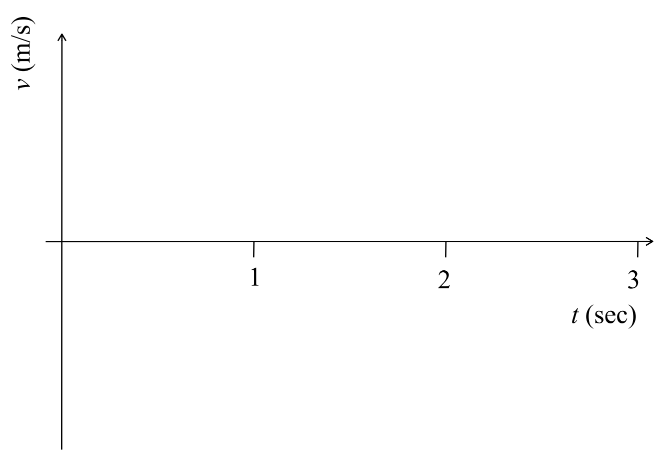
Show your work

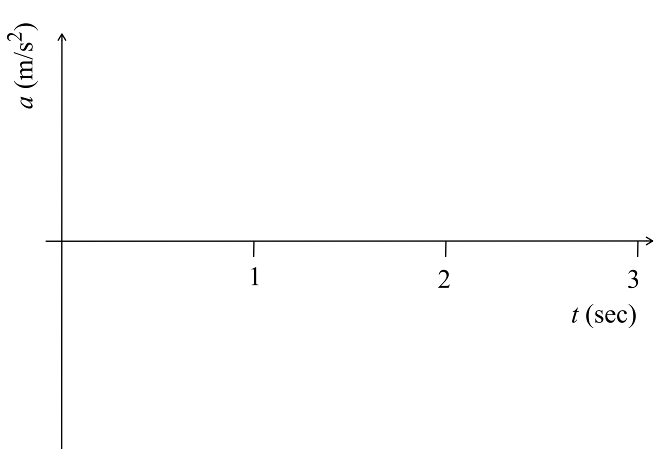
Credit depends on the quality and clarity of your explanations

1. A boy standing on the ground throws a ball directly upwards, and it lands in his hand 3 seconds later. In what follows, use $|g| = 10$ m/s^2, and let the upward direction be positive, measured from the boy's hand. Let the instant the ball was thrown be t = 0 seconds.

   A) Calculate the speed at which the ball was thrown upwards.
   B) Calculate the maximum height above the boy's hand to which the ball rises.
   C) On the axes below, sketch a graph of the position (x) of the ball as a function of time for the three seconds during which the ball is in the air. Clearly show the scale for the vertical (x) axis.
   D) On the axes below, sketch a graph of the velocity (v) of the ball as a function of time for the three seconds during which the ball is in the air. Clearly show the scale for the vertical (v) axis.
   E) On the axes below, sketch a graph of the acceleration (a) of the ball as a function of time for the three seconds during which the ball is in the air. Clearly show the scale for the vertical (a) axis.







2. The speed of an Olympic sprinter in the 100 meter dash is approximated closely by the expression

$$v = A\left(1 - e^{-Bt}\right)$$

where t is in seconds, v is in meters per second, and A and B are constants.

A) What are the SI units of A and B?
B) Determine the initial speed of the sprinter.
C) Determine the initial acceleration of the sprinter.
D) Determine the distance covered by the sprinter in time t.

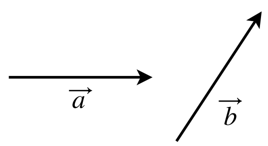
AP Physics C
Unit 2

Name____

Vectors

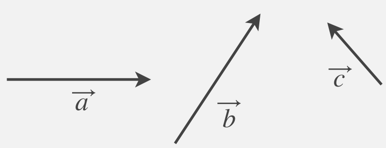
Vectors are used to represent quantities that are characterized by a magnitude (a numerical value with appropriate units) and a direction. The usual example is the *displacement* vector. A quantity with only magnitude and no direction is called a *scalar*. Vectors are symbolized here with a superposed arrow over italic type (e.g., $\vec{a}$). The arrow is the best way to symbolize a handwritten vector. Other texts may use a "harpoon" or boldface type (e.g., $\overrightharpoon{a}$ or **a**) to represent vectors. The scalar magnitude of the vector is symbolized using absolute value signs (e.g., $\left|\vec{a}\right|$) or simply by the letter itself (e.g., *a*).

We agree for the time being that a vector can be moved as long as its length and direction are preserved. To add two vectors, place the tail of the second at the tip of the first. The *vector sum*, or *resultant*, is the vector from the tail of the first to the tip of the second. The set of vectors is said to be *closed* under vector addition; that is, the sum of two vectors is again a vector. In the space below, sketch the vectors $\vec{a} + \vec{b}$ and $\vec{b} + \vec{a}$ to show that vector addition is *commutative* ($\vec{a} + \vec{b} = \vec{b} + \vec{a}$).



$\vec{a} + \vec{b}$ | $\vec{b} + \vec{a}$

In the space below, sketch the vectors $\vec{a} + \left(\vec{b} + \vec{c}\right)$ and $\left(\vec{a} + \vec{b}\right) + \vec{c}$ to show that vector addition is *associative* ($\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$).



$\vec{a} + \left(\vec{b} + \vec{c}\right)$ | $\left(\vec{a} + \vec{b}\right) + \vec{c}$

The *identity element* for vector addition is that vector which, when added to an arbitrary vector $\vec{a}$, yields the vector $\vec{a}$ itself. The identity element for vector addition is called the *zero vector*. It has no length and an undefined direction.

For a given vector $\vec{a}$, the *additive inverse* is that vector which, when added to $\vec{a}$, yields the identity element (the zero vector). We symbolize this vector as $-\vec{a}$. Describe the length and direction of $-\vec{a}$ relative to $\vec{a}$:

(A set of mathematical objects (e.g., vectors) together with an operation (e.g., vector addition), that obeys the rules of closure, associativity, identity and inverse is called a *group*. A group that also obeys commutativity is called an *Abelian* group after the mathematician Niels Abel. Groups are fundamental to the study of elementary particles.)

Can two vectors whose magnitudes are different be added to give a zero resultant? Why or why not?

Can three vectors whose magnitudes are different be added to give a zero resultant? Why or why not?

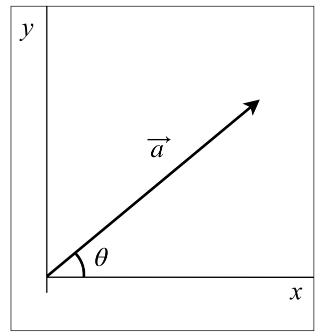
Can the sum of the magnitudes of two vectors ever be equal to the magnitude of the sum of those vectors? Explain.

What are the properties of two vectors $\vec{a}$ and $\vec{b}$ such that

A) $\vec{a} + \vec{b} = \vec{c}$ and $a + b = c$?

B) $\vec{a} + \vec{b} = \vec{c}$ and $a^2 + b^2 = c^2$?

Components and Unit Vectors



A given vector can be expressed as the sum of two other vectors in an infinite variety of ways. Given a coordinate system, it is useful to express vectors in terms of the sum of two vectors, each of which points parallel to an axis. These are called the *vector components* of the original vector. For a vector $\vec{a}$ in the *x-y* plane, its vector components are called $\vec{a}_x$ and $\vec{a}_y$. Draw these vectors in the space to the right.

The *scalar components* or just *components* of $\vec{a}$ are just the magnitudes $\left|\vec{a}_x\right|$ and $\left|\vec{a}_y\right|$.

Express the following in terms of a and θ: $a_x =$ ____ $a_y =$ ____

Express the following in terms of a_x and a_y: $a =$ ____ $\theta =$ ____

Vector equations in n dimensions are shorthand for n scalar equations at once. For example, the vector equation $\vec{a} = \vec{b} + \vec{c}$ in three dimensions is equivalent to the three scalar equations $a_x = b_x + c_x$, $a_y = b_y + c_y$, and $a_z = b_z + c_z$.

In a 3-D Cartesian coordinate system, we define *unit vectors* in the x, y, and z directions as vectors with magnitude 1. We call them $\hat{\imath}$ (spoken "i-hat"), $\hat{\jmath}$, and $\hat{k}$, respectively. The circumflex or "hat" indicates that they have magnitude 1. Other conventions exist, such as $\hat{x}$, $\hat{y}$, and $\hat{z}$; or $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$; or $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$**.** Given an arbitrary vector $\vec{a}$, express it in terms of the magnitudes of its components and the unit vectors:

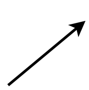
$\vec{a} =$ ____

Can a vector have zero magnitude if one of its components is non-zero? Explain.

Multiplication of Vectors

There are three kinds of vector multiplication: *Scalar multiplication*, *dot product*, and *cross product*.

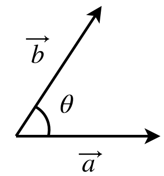
1. Scalar multiplication is the multiplication of a vector by a scalar. The result is a vector collinear with the original vector. If the scalar is a pure number, then the result is a vector of the same kind as the original, but if the scalar has units, then the result is a vector of a different type. Given the vector $\vec{v}$ below, with magnitude $v = 2\,\mathrm{m/s}$ draw the vectors that result from the scalar multiplication by the given scalar, and indicate the magnitude of each.



scalar: –2; magnitude: ____________ scalar: 3 s; magnitude: ____________

Note that the vector you draw in the second box does *not* have to be three times as long as $\vec{v}$. The scales of vectors of different types are independent of each other.

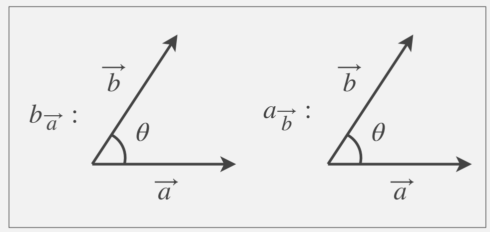
2. The dot product of two vectors $\vec{a}$ and $\vec{b}$ is symbolized by $\vec{a} \cdot \vec{b}$. It is the scalar $a\,b\,\cos\theta$, where θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$ when placed tail-to-tail. Is the set of vectors closed under this operation? Why or why not?



Is the dot product commutative? Why or why not?

Is the dot product associative? Why or why not?

Consider the following statement: "$\vec{a} \cdot \vec{b} = a\,b_{\vec{a}} = a_{\vec{b}}\,b$ where, for example, $b_{\vec{a}}$ means the length of the vector component of $\vec{b}$ in the direction of $\vec{a}$." In the space to the right, show how you can find $b_{\vec{a}}$ and $a_{\vec{b}}$. Show why the statement follows from the definition of dot product.



Consider varying the angle θ between two vectors $\vec{a}$ and $\vec{b}$. What is the angle that makes $\vec{a} \cdot \vec{b}$ the greatest?

$\theta =$ ____ Then $\vec{a} \cdot \vec{b} =$ ____ What angle makes $\vec{a} \cdot \vec{b}$ the least? $\theta =$ ____ Then $\vec{a} \cdot \vec{b} =$ ____

If $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, does it follow that $\vec{b} = \vec{c}$? Why or why not?

Explain how one could use the dot product to find the angle between two vectors.

Consider the table to the right, listing all possible dot products of the unit vectors. Of the 9 dot products, how many are zero? Fill these in and explain why.

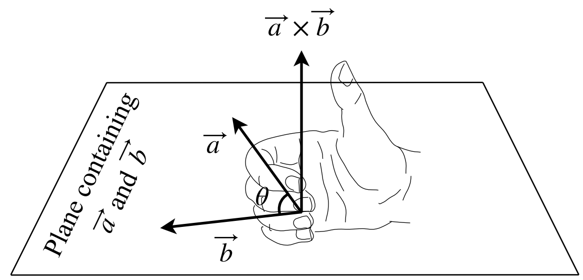
$\hat{\imath} \cdot \hat{\imath} =$	$\hat{\jmath} \cdot \hat{\jmath} =$	$\hat{k} \cdot \hat{k} =$
$\hat{\imath} \cdot \hat{\jmath} =$	$\hat{\jmath} \cdot \hat{k} =$	$\hat{k} \cdot \hat{\imath} =$
$\hat{\imath} \cdot \hat{k} =$	$\hat{\jmath} \cdot \hat{\imath} =$	$\hat{k} \cdot \hat{\jmath} =$

What is the value of the remaining dot products in the table? Fill these in and explain why. (Are these values vectors or scalars?)

The dot product of $\vec{a}$ and $\vec{b}$ can be expressed in terms of the unit vectors as $\vec{a} \cdot \vec{b} = \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right) \cdot \left(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}\right)$. The distributive law holds for dot products, giving a total of 9 terms when the product is written out. Of the 9 terms, how many are non-zero? Explain.

Express the dot product of $\vec{a}$ and $\vec{b}$ in terms of their six components: $\vec{a} \cdot \vec{b} =$

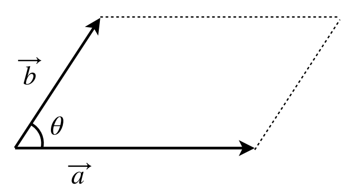
3. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is a vector symbolized by $\vec{a} \times \vec{b}$. Its magnitude is $a\,b\,\sin\theta$, where θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$ when placed tail-to-tail. The direction of $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, and is found by the *right-hand rule*: Curl the fingers of your right hand from $\vec{a}$ to $\vec{b}$. Your thumb points in the direction of $\vec{a} \times \vec{b}$. Is the set of vectors closed under this operation? Why or why not? (Hint: What if the vectors are not dimensionless?)



Is the cross product commutative? Why or why not?

Is the cross product associative? Why or why not? (Hint: Consider three different vectors in the same plane.)

Consider the two vectors $\vec{a}$ and $\vec{b}$ shown tail-to-tail to the right with angle θ between them. We complete the parallelogram as shown in the diagram. What does the magnitude of the vector $\vec{a} \times \vec{b}$ have to do with this parallelogram? Explain.



Consider varying the angle θ between two vectors $\vec{a}$ and $\vec{b}$. What is the angle that makes $\vec{a} \times \vec{b}$ the greatest?

$\theta =$ Then $\left|\vec{a} \times \vec{b}\right| =$. What angle makes $\vec{a} \times \vec{b}$ = the least? $\theta =$ Then $\left|\vec{a} \times \vec{b}\right| =$

Consider the table to the right, listing all possible cross products of the unit vectors. Of the 9 cross products, how many are zero? Fill these in and explain why.

$\hat{i} \times \hat{i}$	$\hat{j} \times \hat{i}$	$\hat{k} \times \hat{i}$
$\hat{i} \times \hat{j}$	$\hat{j} \times \hat{j}$	$\hat{k} \times \hat{j}$
$\hat{i} \times \hat{k}$	$\hat{j} \times \hat{k}$	$\hat{k} \times \hat{k}$

What is the *magnitude* of the remaining cross products? Explain.

Use the right-hand rule to fill in the remaining cross products in the table, using unit vector notation.

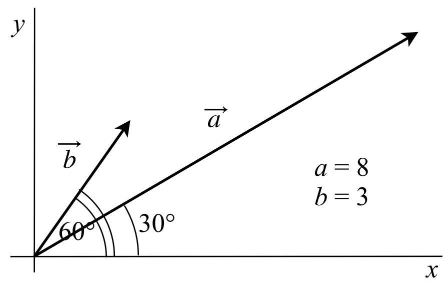
The cross product of $\vec{a}$ and $\vec{b}$ can be expressed in terms of the unit vectors as $\vec{a} \times \vec{b} = \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right) \times \left(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}\right)$. The distributive law holds for cross products, giving a total of 9 terms when the product is written out. Of the 9 terms, how many are non-zero? Explain.

Express the cross product of $\vec{a}$ and $\vec{b}$ in terms of their six components and the unit vectors:

$\vec{a} \times \vec{b} =$

Examples

1. Given the vectors $\vec{a}$ and $\vec{b}$ shown to the right, express the following in terms of the unit vectors (where appropriate). Assume that the positive z direction is out of the page.



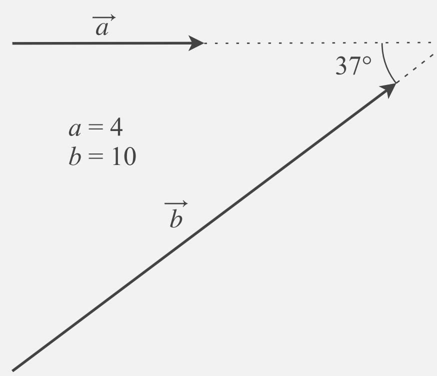
A) the vector $\vec{a}$:

B) the vector $\vec{b}$:

C) the product $\vec{a} \cdot \vec{b}$:

D) the product $\vec{a} \times \vec{b}$:

2. Given the vectors $\vec{a}$ and $\vec{b}$ shown to the right, express the following in terms of the unit vectors (where appropriate). Assume that the positive z direction is out of the page.



A) $\vec{a} + \vec{b}$:

B) $\vec{a} - \vec{b}$:

C) $\vec{b} - \vec{a}$:

D) $\vec{a} \cdot \vec{b}$:

E) $\vec{a} \times \vec{b}$:

F) $\vec{b} \times \vec{a}$:

3. Simplify the following products:

A) $-\hat{j} \cdot \hat{i} =$

B) $\hat{i} \times -\hat{k} =$

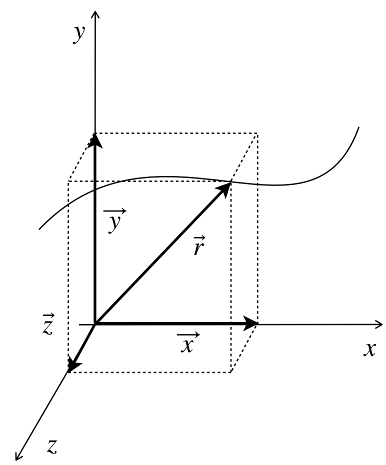
C) $\hat{j} \cdot \left(\hat{i} \times \hat{k}\right) =$

4. Given $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} - \hat{k}$ and $\vec{b} = 2\hat{\imath} + 4\hat{k}$, find the magnitude of the vector product $\vec{a} \times \vec{b}$.

5. Determine the value of n so that $\vec{a} = 2\hat{\imath} + n\hat{\jmath} + \hat{k}$ is perpendicular to $\vec{b} = 4\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$.

6. Find the angle between the vectors $\hat{\imath} + \hat{\jmath} + \hat{k}$ and $\hat{\imath} - \hat{\jmath} - \hat{k}$.

Particle Kinematics in 2 and 3 Dimensions



In the one-dimensional case, we took position (x) and time (t) as fundamental quantities, defined velocity (v) and acceleration (a) in terms of time derivatives, and worked out the kinematics by integration. In multiple dimensions, the difference is that position is now a vector ($\vec{r}$) with components (e.g., $\vec{x}$, $\vec{y}$,and $\vec{z}$ in three dimensions). We define the velocity vector as the derivative of the position vector:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

As always, this vector equation stands for three scalar equations at once:

$v_x =$ ☐, $v_y =$ ☐, and $v_z =$ ☐.

If $\vec{r}$ has a constant length, is it possible for $\frac{d\vec{r}}{dt}$ to be nonzero? Explain.

We define the acceleration vector as the derivative of the velocity vector: $\vec{a} =$ ☐

If acceleration is constant (both in magnitude and in direction), then the kinematic equations in vector form can be derived in a manner similar to the one-dimensional case. Write the definition of acceleration as a differential equation: ☐

Choose appropriate limits and write the integral (limits can be vectors or scalars, as appropriate):

Integrate to get the equation that corresponds to $v = v_o + at$.

Write this as a differential equation in $\vec{r}$:

Then integrate again to get the equation that corresponds to $x = x_o + v_o t + \frac{1}{2}at^2$.

Now combine these to get the vector equations that correspond to the other two. Eliminate $\vec{a}$ to get the equation that corresponds to $x = x_o + \left(\frac{v + v_o}{2}\right)t$:

Then take the dot product of the first equation with itself, and use the distributive law and the second equation to put it in the form that corresponds to $v^2 = v_o^2 + 2a\left(x - x_o\right)$:

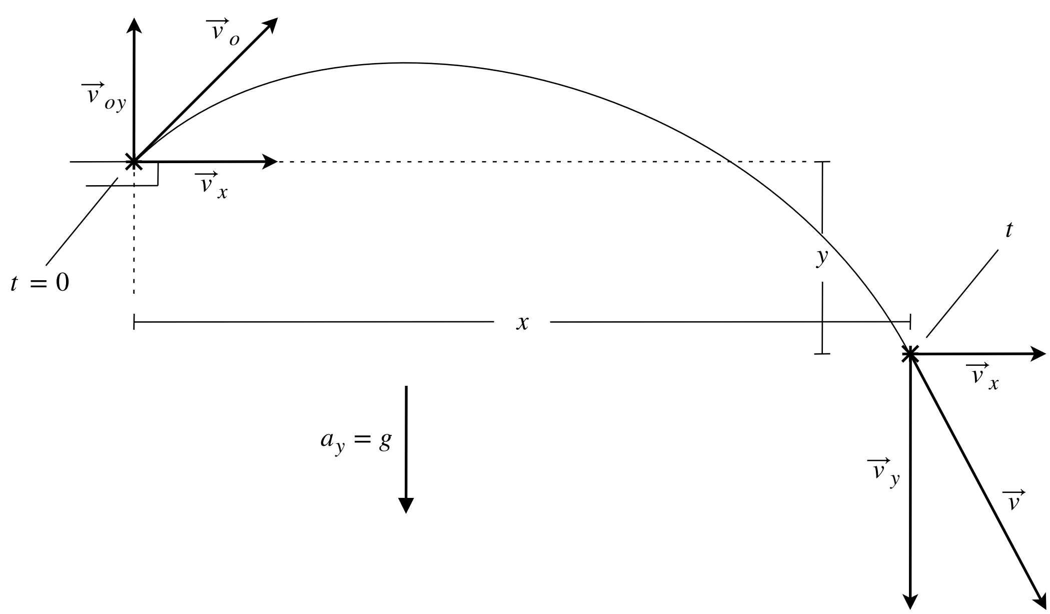
Recall from Unit 1 (page 1.2) that we restricted our study of kinematics to rigid, non-rotating objects in one dimension. We now consider examples of two-dimensional motion; the extension to three (or more) dimensions is straightforward once we understand vector notation.

Projectile Motion

Neglecting air resistance, a particle projected near the surface of the earth undergoes two simultaneous motions: In the vertical plane (the y-direction), it undergoes motion with constant acceleration ($a_y = g$, down). Horizontally (the x-direction) it moves with constant velocity ($a_x = 0$), so we only need to consider three horizontal variables: x, v_x, and t. The four kinematic equations simplify to one: $x = v_x t$.

Projectile motion occurs during a *time interval*, say from 0 to t. The variables that describe the motion are given in the following table:

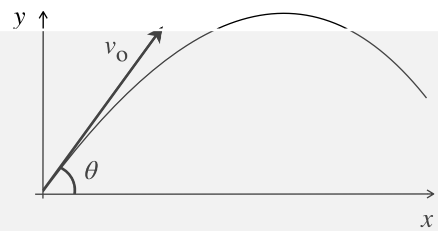
Vertical		**Horizontal**	
Variable	**Meaning**	**Variable**	**Meaning**
y	The *net vertical* displacement from the beginning to the end of the interval	x	The *net horizontal* displacement from the beginning to the end of the interval.
a_y	Always g, the acceleration due to gravity, directed down		
v_{oy}	The *vertical component* of the velocity at the beginning of the interval (at $t = 0$).	v_x	The constant horizontal velocity throughout the interval
v_y	The *vertical component* of the velocity at the end of the interval (at $t = t$).		
t	The elapsed time from the beginning to the end of the interval	t	The elapsed time from the beginning to the end of the interval



Solving projectile motion problems amounts to setting up tables for vertical and horizontal motions like the one to the right, and noting that, since the motion is *simultaneous*, the t in both columns is the same.

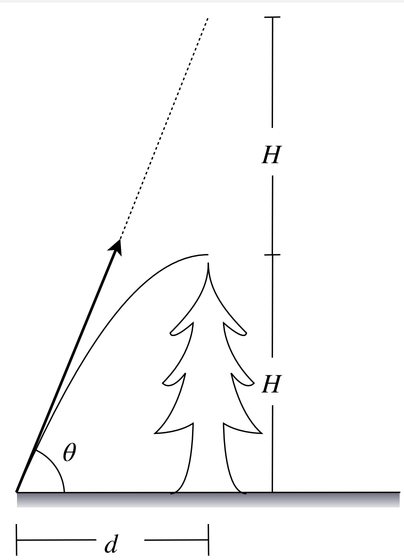
vert	horiz

Examples

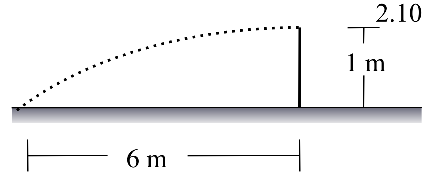


1. A projectile is launched from the origin of the x-y plane with an initial speed v_o at an angle θ from the horizontal. Show that y is a quadratic function of x, so that the trajectory is a parabola.

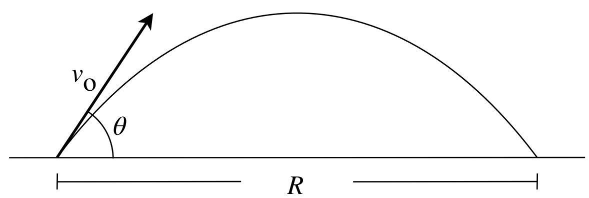
2. A golf ball is at a distance d from the base of a tree of height H as shown to the right. Show that in order to chip the ball so that it just clears the tree at the peak of its flight, the golfer must choose a club whose loft aims the ball at a point directly above the tree, twice the height of the tree. (ie., show that $\tan\theta = \dfrac{2H}{d}$)



3. A tennis player half-volleys a ball from a point 6 m from the net, and it just clears the net (which is 1 m high at that point) at the peak of its flight. (A half-volley occurs when the ball is struck just after it hits the ground.) Determine the initial velocity and the initial angle at which the ball was half-volleyed.



4. A cannon shoots a projectile on level ground at an initial speed v_o and angle θ as shown to the right.



A) Determine the range R of the projectile.

B) Show that the maximum range R_{max} is achieved when $\theta = 45°$.

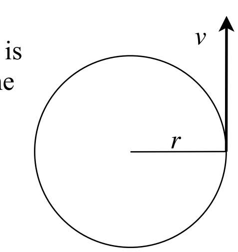
For a projectile shot at 45°, determine the following in terms of R_{max} and constants:

C) the maximum height reached

D) the time of flight

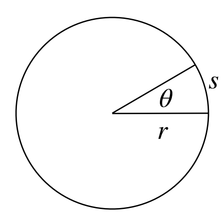
Uniform Circular Motion

A particle moving in a circular path of radius r at a constant speed v is in *uniform circular motion*. Note that the speed v is constant but the velocity $\vec{v}$ is not, since its direction (tangent to the circle at all times) is not. If the time it takes for the particle to go completely around the circle is T (called the *period*), then the

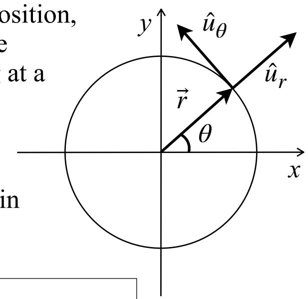


speed is: $v =$

Consider a time t, shorter than T. Let s represent the distance along the arc that the particle travels in time t. Evidently the speed v is the rate of change with respect to time of arc length s. Express s in terms of v and t and simplify: Let θ be the angle subtended by s at the center of the circle. Express s in terms of r and θ: Combine these two equations to eliminate s: Solve this equation for θ: Take the derivative with respect to t: $\dfrac{d\theta}{dt} =$ This *rate of change of angle* is called the *angular speed* ω.



To completely describe the kinematics of uniform circular motion, we need expressions for the position, velocity and acceleration vectors of the particle at any time. Using polar coordinates makes these expressions simpler, but requires the definition of *polar unit vectors*. Consider a particle moving at a constant speed v in a circle of radius r. Its polar coordinates are (r, θ).

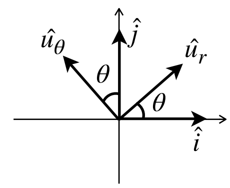


We define the polar unit vectors as vectors of length 1; $\hat{u}_r$ points in the same direction as $\vec{r}$ (radially out from the center of the circle) and $\hat{u}_\theta$ points perpendicular to $\hat{u}_r$, tangent to the circle in the same direction as θ increases (ie., counterclockwise).

Express the position vector in terms of polar unit vectors and the appropriate scalar: $\vec{r} =$

Express the velocity vector in terms of polar unit vectors and the appropriate scalar: $\vec{v} =$

Unlike the Cartesian unit vectors $\hat{i}$ and $\hat{j}$, which are constant, the polar unit vectors change direction with time, so they are not constant. They therefore have nonzero derivatives. Use the diagram to the right to express $\hat{u}_r$ and $\hat{u}_\theta$ in terms of the Cartesian unit vectors and the angle θ:



$\hat{u}_\theta =$ $\hat{u}_r =$

Find the time derivative of $\hat{u}_\theta$. You will need to use the chain rule since θ is a function of time:

$$\frac{d\hat{u}_\theta}{dt} =$$

Use the definition of angular velocity to express this derivative in terms of v, r, and polar unit vectors: $\frac{d\hat{u}_\theta}{dt} =$

We're now ready to express the acceleration vector in terms of polar unit vectors and the appropriate scalars. Recall that v is constant, but $\vec{v}$ is not: $\vec{a} = \frac{d\vec{v}}{dt} =$

What does this expression say about the *magnitude* and the *direction* of the acceleration in uniform circular motion?

The General Case of Two-Dimensional Motion

The general case of any motion whatsoever in two dimensions can be worked out based on the expressions above for uniform circular motion. We have restricted the particle to constant speed v and constant radius r. If we relax these two restrictions, we have a particle that can change its speed, and change the radius of curvature of its path; this is general enough to describe *any* two-dimensional motion.

Express the acceleration vector if the speed v isn't constant. You will need to use the product rule:

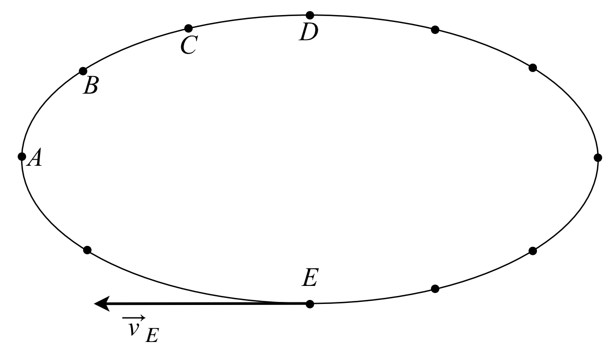
$$\vec{a} = \frac{d\vec{v}}{dt} =$$

This expression breaks the acceleration vector into two components, one centripetal (center-seeking) and one tangential. Recall that acceleration means rate of change of velocity, and that the velocity vector can change in two ways: magnitude and direction. The centripetal component doesn't change the magnitude (the speed), only the direction. The tangential component doesn't change the direction, only the speed.

In the general case, we relax the restriction that the particle moves in a circle of constant radius. We define the *instantaneous radius of curvature* of a path as the radius of the circle that matches the path over a differentially short arc. The "r" in the expression on the previous page is then interpreted as the instantaneous radius of curvature of the particle's path.

Examples

1. An object is moving along an oval track at *constant speed* as shown to the right. The dots represent successive positions at equal time intervals. The velocity vector for the object is drawn at point E.



A) On the diagram, draw velocity vectors at points A - D consistent with the given vector $\vec{v}_E$.

B) For the time interval from B to C, construct the vector representing the change in velocity $\Delta\vec{v}_{BC}$

C) Imagine the point C moving closer to point B. As it does so, how does the direction of $\Delta\vec{v}_{BC}$ change?

D) In the limit, as point C approaches point B, the $\Delta\vec{v}_{BC}$ vector becomes the $d\,\vec{v}_B$ vector. What is the direction of this vector compared to the $\vec{v}_B$ vector?

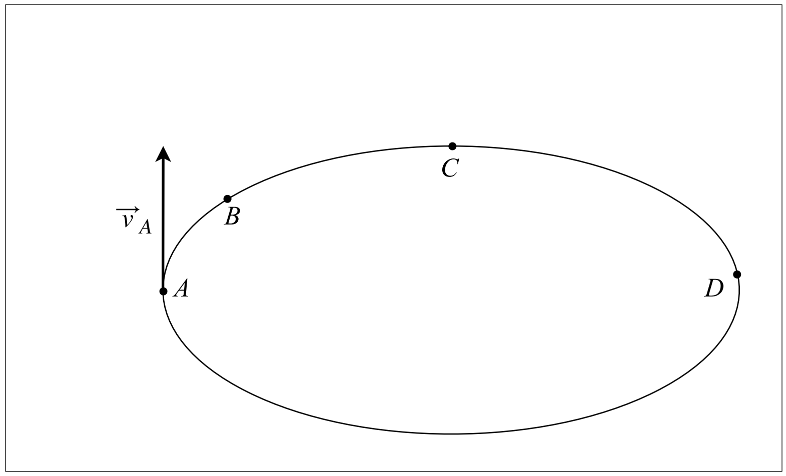
E) On the diagram, draw a vector representing the acceleration of the object at point B. Explain how you know its direction.

F) On the diagram, construct the vector representing $\Delta\vec{v}_{AB}$. How does the length of $\Delta\vec{v}_{AB}$ compare to the length of $\Delta\vec{v}_{BC}$?

G) On the diagram, draw a vector representing the acceleration at A, consistent with your $\vec{a}_B$ vector.

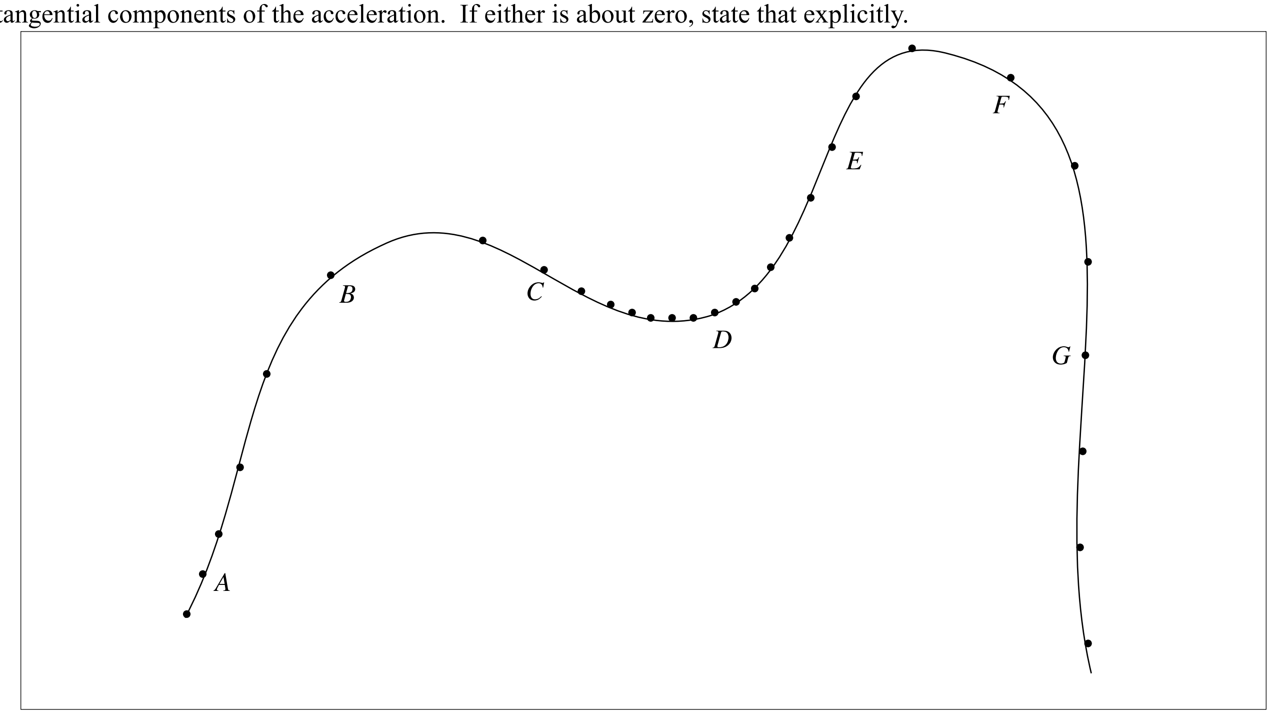
H) Generalize the discussion above to make a statement about the behavior of the acceleration vector of a particle moving at constant speed along a curve with a varying radius of curvature.

2. An object is moving along an oval track at *increasing speed* as shown to the right. The dots represent successive positions at equal time intervals. The velocity vector for the object is drawn at point A.
   A) Draw velocity vectors at points B - D consistent with the given vector $\vec{v}_A$.
   B) For the time interval from B to C, construct the vector representing the change in velocity $\Delta\vec{v}_{BC}$
   C) Imagine the point C moving closer to point B. As it does so, how does the direction of $\Delta\vec{v}_{BC}$ change?



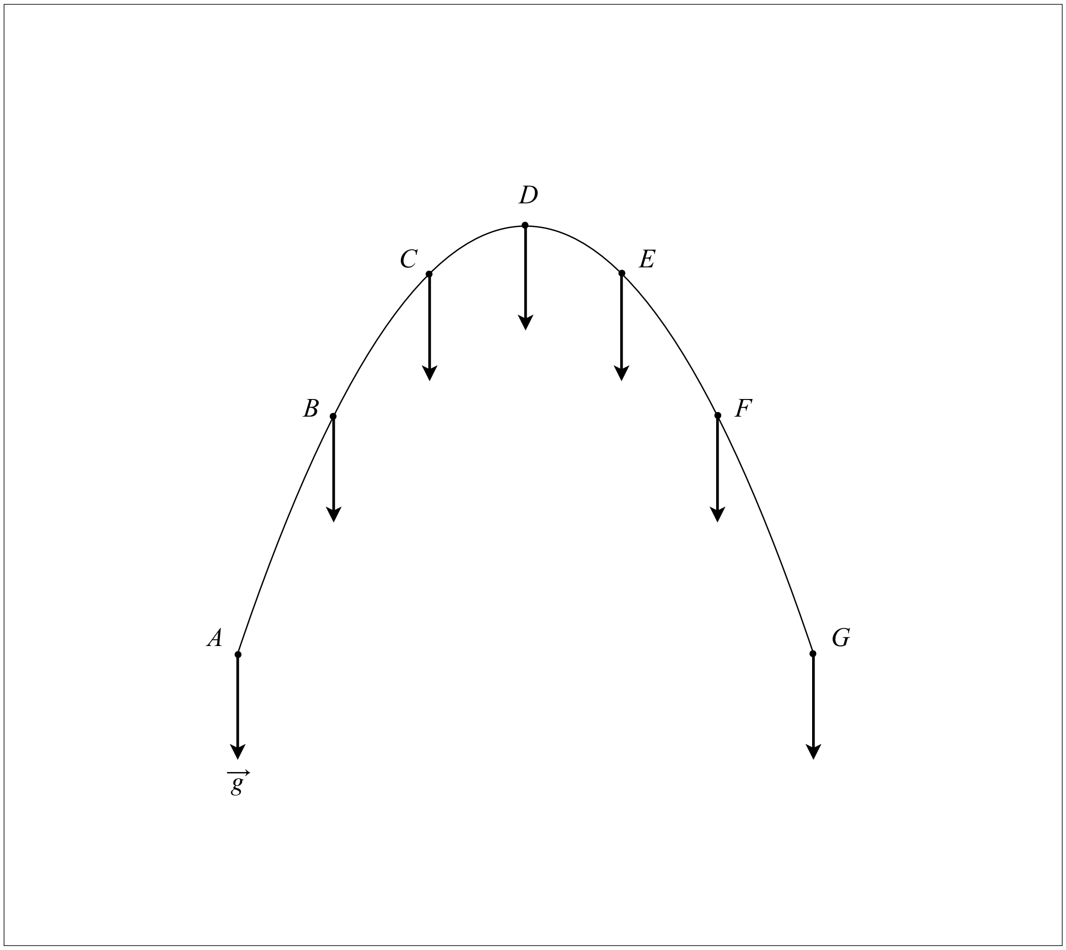
   D) In the limit, as point C approaches point B, the $\Delta\vec{v}_{BC}$ vector becomes the $d\vec{v}_B$ vector. What is the direction of this vector compared to the $\vec{v}_B$ vector? How is it different from the previous example?
   E) On the diagram, draw a vector representing the acceleration of the object at point B.
   F) Make a general statement about a particle moving at a *varying* speed along a given path, compared to a particle moving with *constant* speed along the same path.

3. Below is a curve representing the trajectory of a particle in two dimensions. The dots represent successive positions of the particle at equal time intervals. At the lettered points A - G, draw reasonable vectors representing the centripetal and tangential components of the acceleration. If either is about zero, state that explicitly.



4. Below is a parabola representing the trajectory of a projectile. Each dot represents successive positions at equal time intervals. The acceleration at every point on the parabola has the same magnitude and direction; it is $\vec{g}$, the acceleration due to gravity, and the diagram shows these vectors.

   A) Draw vectors at each point representing the centripetal and tangential components of the acceleration ($\vec{a}_c$ and $\vec{a}_t$), consistent with the given vectors.



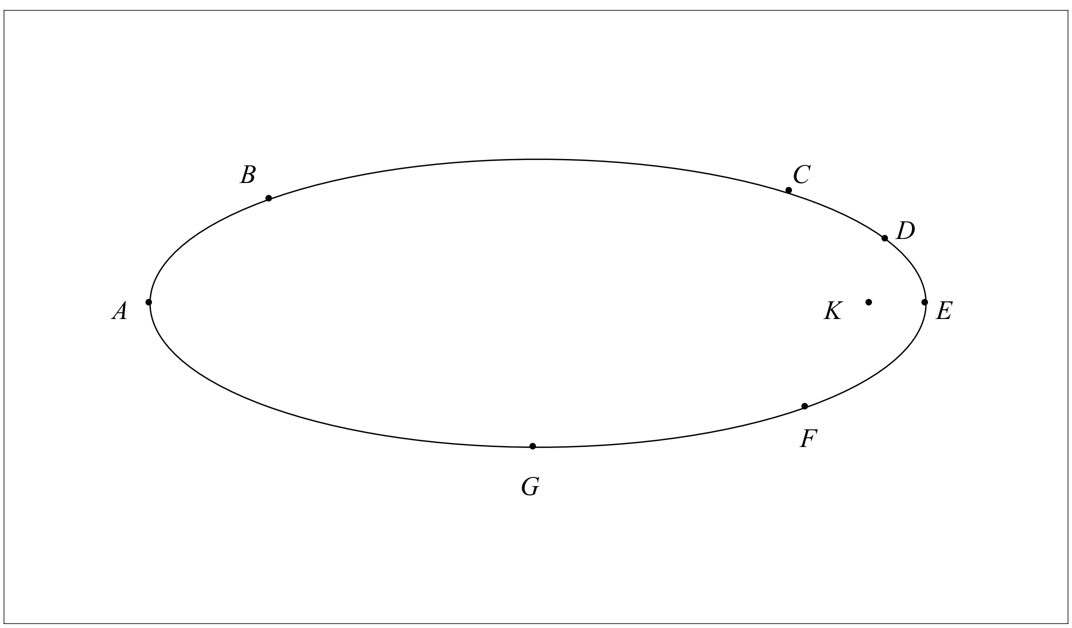
   B) Using the terms "slowing down" and "speeding up," describe the motion of the projectile from *A* to *G* in terms of the tangential component of the acceleration.

   C) Using the term "radius of curvature," describe the motion of the projectile from *A* to *G* in terms of the centripetal component of the acceleration.

	Instant 1	Instant 2	Instant 3	Instant 4	Instant 5
Acceleration	→	↓	↑	↑	↑
Velocity	→	↑	↖	←	↙
Description	☐ Speeding up ☐ Slowing down ☐ Constant speed	☐ Speeding up ☐ Slowing down ☐ Constant speed	☐ Speeding up ☐ Slowing down ☐ Constant speed	☐ Speeding up ☐ Slowing down ☐ Constant speed	☐ Speeding up ☐ Slowing down ☐ Constant speed

5. Each diagram above shows acceleration and velocity vectors for an object at different instants in time. For each instant, check the box corresponding to the correct description. Then write a general statement below using the concept of "dot product" that summarizes the reasoning you used to check the boxes.

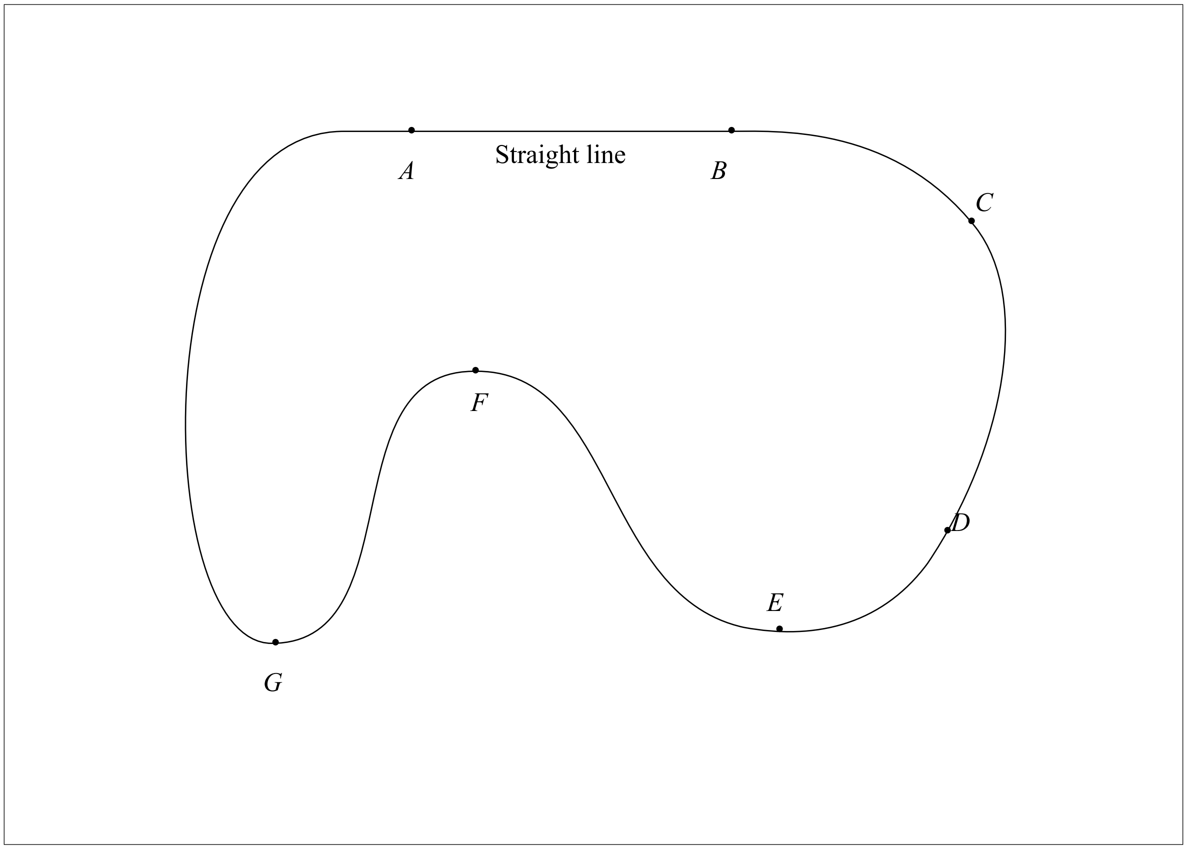
6. An object moves clockwise along the trajectory shown below, looking down from the top. The acceleration varies, but is always directed at point *K*.



A) Draw arrows on the diagram at points *A* – *G* to indicate the direction of the acceleration at each point.
B) For each point, check whether the object is speeding up, slowing down, or neither.

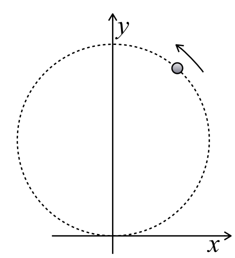
A	*B*	*C*	*D*	*E*	*F*	*G*
☐ Speeding up ☐ Slowing down ☐ Neither	☐ Speeding up ☐ Slowing down ☐ Neither	☐ Speeding up ☐ Slowing down ☐ Neither	☐ Speeding up ☐ Slowing down ☐ Neither	☐ Speeding up ☐ Slowing down ☐ Neither	☐ Speeding up ☐ Slowing down ☐ Neither	☐ Speeding up ☐ Slowing down ☐ Neither

7. An object moves clockwise once around the track shown to the right, looking down from the top. Starting from rest at point A, it speeds up at a constant rate until just past point C, and by the time it reaches point D it is traveling at a constant speed. It then travels at a constant speed the rest of the way around the track.



A) On the diagram, draw acceleration vectors at each point; if the acceleration is zero, state that explicitly.
B) How does the magnitude of the acceleration at E compare to that at G? Explain.

8. A particle travels with a constant speed counterclockwise in a circle of radius 3 m as shown to the right, completing one revolution in 20 s. The bottom of the circle is chosen as the origin of the coordinate system, where the particle starts at $t = 0$. Determine the following in terms of the unit vectors $\hat{i}$ and $\hat{j}$:



A) The position vector at $t = 5$ s

B) The position vector at $t = 7.5$ s

C) The particle's displacement vector during the interval from $t = 5$ s to $t = 10$ s

D) The particle's average velocity vector for the interval from $t = 5$ s to $t = 10$ s

E) The particle's velocity vector at $t = 5$ s

F) The particle's velocity vector at $t = 10$ s

G) The particle's acceleration vector at $t = 5$ s

H) The particle's acceleration vector at $t = 10$ s

AP Physics C
Unit 2 Practice Test

Name ___________________________

Part I
Multiple Choice

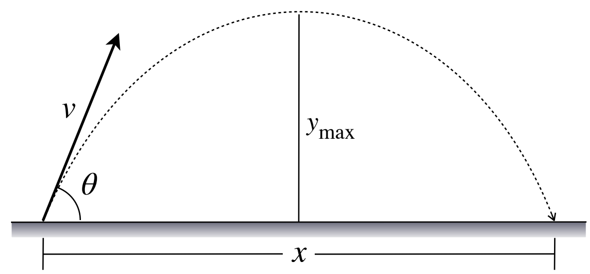
1. A ball is thrown horizontally from the top of a tower 40 m high. The ball strikes the ground at a point 80 m from the bottom of the tower. The angle that the velocity vector makes with the horizontal just before the ball hits the ground is

   A) 45°
   B) 41°
   C) 0°
   D) 90°
   E) 82°

2. A cannon fires a shell at a target 1,200 m away on a horizontal, flat plain. (Ignore air resistance.) While the shell is in flight, its acceleration is

   A) nonzero and uniform, only vertical
   B) nonzero and uniform, both vertical and horizontal
   C) time-varying, only vertical
   D) time-varying, only horizontal
   E) time-varying, both vertical and horizontal

For questions 3 and 4: As shown in the diagram to the right, a batter hits a fly ball with speed v and angle θ from the horizontal. Ignore air resistance and the height of the batter.



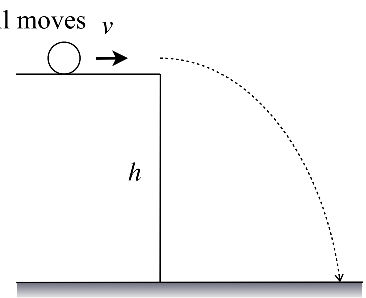
3. The maximum height that the ball will reach (y_{max}) is

   A) $\dfrac{(v \sin \theta)^2}{2g}$
   B) $\dfrac{v^2}{2g}$
   C) $\dfrac{(v \sin \theta)^2}{g}$
   D) $9.8\,v$
   E) $\dfrac{v \sin \theta}{g}$

4. The horizontal distance x covered by the ball is

   A) $\dfrac{2v^2}{g} \sin \theta$ B) $\dfrac{2v^2}{g}$ C) $\dfrac{v^2}{g} \sin(2\theta)$ D) $\dfrac{v^2}{g} \cos \theta$ E) vg

5. A ball rolls off a table of height h at a horizontal speed v, as shown to the right. The ball moves as a projectile until it hits the floor. Neglecting air resistance, which quantity remains constant while the ball is a projectile?



   I. its horizontal speed
   II. its vertical speed
   III. its vertical acceleration

   A) I only B) II only C) III only D) I and III E) II and III

6. The velocity of a projectile equals its initial velocity added to

   A) a constant horizontal velocity
   B) a constant vertical velocity
   C) a constantly increasing horizontal velocity
   D) a constantly increasing downward velocity
   E) a constant velocity directed at the target

7. Identical guns fire identical bullets horizontally at the same speed from the same height above level plains, one on the earth and one on the moon. Which of the following statements is/are true?

   I. The horizontal distance traveled by the bullet is greater for the moon.
   II. The flight time is less for the bullet on the earth.
   III. The velocity of the bullets at impact are the same.

   A) III only B) I and II C) I and III D) II and III E) I, II, and III

8. A bomber flying in level flight at constant velocity must release its bomb before it is over its target. Neglecting air resistance, which one of the following is *not necessarily* true?

   A) The bomber will be over the target when the bomb strikes.
   B) *g* remains constant for the bomb.
   C) The horizontal velocity of the plane equals the vertical velocity of the bomb when it hits the target.
   D) The bomb travels in a curved path.
   E) The time of flight of the bomb is independent of the horizontal speed of the plane.

9. An object is shot from the back of a truck moving at 30 mph on a straight horizontal road. The gun is aimed upward, perpendicular to the bed of the truck. Neglecting air resistance, the object falls

   A) in front of the truck
   B) behind the truck
   C) on the truck
   D) in front, behind, or on the truck depending on the initial speed of the object
   E) in front, behind, or on the truck depending on the value of *g*

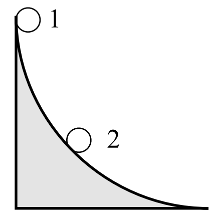
10. A particle moves at constant speed in a circular path. The instantaneous velocity and instantaneous acceleration vectors are

   A) both tangent to the circular path
   B) both perpendicular to the circular path
   C) perpendicular to each other
   D) opposite to each other
   E) none of the above

11. A circus cannon fires an acrobat into the air at an angle of 45° to the horizontal, and the acrobat reaches a maximum height y above her original launch height. The cannon is now aimed so that it fires straight up into the air. The maximum height reached by the same acrobat is now

A) y
B) $y/2$
C) $2y$
D) $\sqrt{2}y$
E) $\dfrac{2y}{\sqrt{2}}$

12. A small ball is released from rest at position 1 and rolls down a vertical circular track under the influence of gravity. When the ball reaches position 2, the direction that most nearly corresponds to the direction of the ball's acceleration is



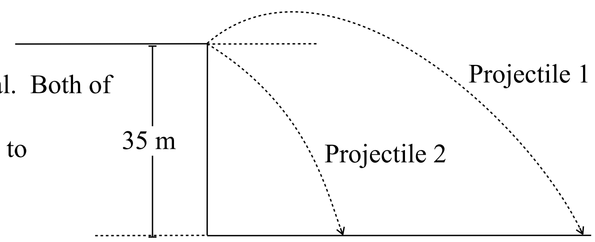
A) $\leftarrow$ B) $\rightarrow$ C) $\nearrow$ D) $\swarrow$ E) $\searrow$

13. An object moves in two dimensions according to

$$\vec{r}(t) = \left(4.0\ \mathrm{m/s^2}\,t^2 - 9.0\ \mathrm{m}\right)\hat{i} + \left(2.0\ \mathrm{m/s}\,t - 5.0\ \mathrm{m}\right)\hat{j}$$

The object crosses the x axis at
A) $t = 0.0$ s
B) $t = 0.4$ s
C) $t = 0.6$ s
D) $t = 1.5$ s
E) $t = 2.5$ s

14. Two projectiles are launched from a 35 meter ledge as shown in the diagram. One is launched from a 37° angle above the horizontal and the other is launched from 37° below the horizontal. Both of the launches are given the same initial speed of $v_o = 50$ m/s. The difference in the times of flight for these two projectiles is closest to



A) 3 s
B) 5 s
C) 6 s
D) 8 s
E) 10 s

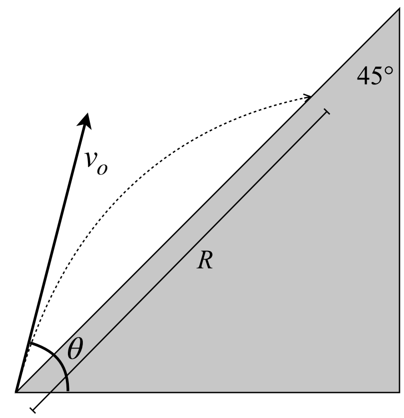
15. A particle starts at rest and travels in a circle of radius 20 cm, speeding up at the rate of 4 cm/s^2. 2.5 seconds after it starts from rest, the angle between the particle's velocity vector and its acceleration vector is

A) zero
B) 39°
C) 51°
D) 90°
E) 129°

1.A 2.A 3.A 4.C 5.D 6.D 7.B 8.C 9.C 10.C 11.C 12.B 13.E 14.C 15.C

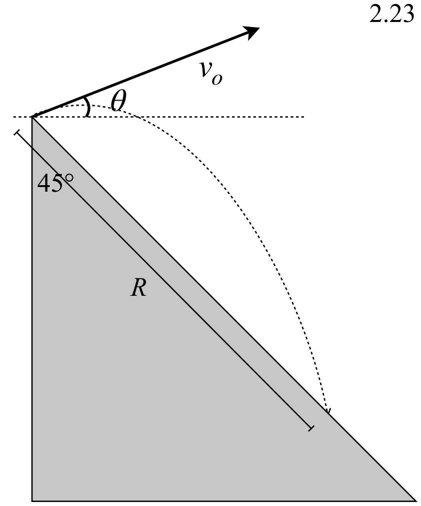
Part II
Show your work
Credit depends on the quality and clarity of your explanations

1. An inclined plane makes an angle of 45° with the horizontal. A projectile is launched at an angle θ with the horizontal with a speed v_o up the incline as shown to the right. Find:



A) the time of flight t of the projectile.
B) the range R of the projectile up the incline.
C) the angle θ for which the range is a maximum. (Hint: The double-angle formulas will be useful: $2\sin\theta\cos\theta = \sin 2\theta$ and $\cos^2\theta - \sin^2\theta = \cos 2\theta$.)

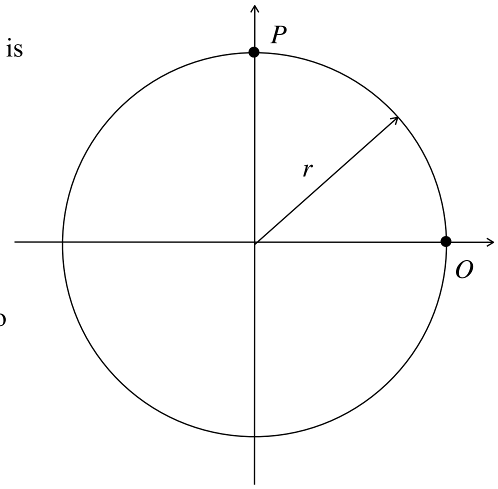
(*continued on next page*)



1. (continued) The projectile is now launched at the same initial speed from the top of a 45° incline at an angle θ with the horizontal as shown below. Modify your solutions above to find:

   D) the time of flight t of the projectile.
   E) the range R of the projectile down the incline.
   F) the angle θ for which the range is a maximum.

2. A particle starts from rest at point O and travels counterclockwise around a circular track of radius r with a tangential acceleration whose magnitude a_t is constant. In terms of r and a_t, find:

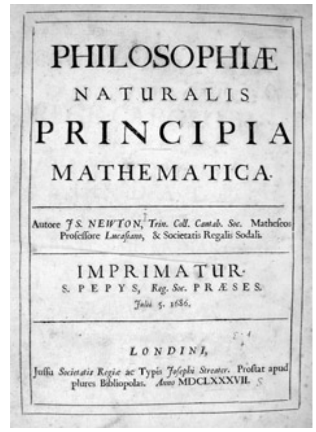


A) the particle's speed v as a function of time t
B) the particle's speed at point P
C) the magnitude of the particle's centripetal acceleration a_c as a function of time t
D) the magnitude of the particle's centripetal acceleration a_c at point P
E) i) how much time it takes for the particle's centripetal acceleration to be equal in magnitude to its tangential acceleration
ii) how far along the circumference of the circle the particle goes during this time.

AP Physics C
Unit 3

Name____

Dynamics



Having considered the pure *description* of motion (kinematics), we now include in the discussion the *causes* of motion (dynamics). What we now call Newtonian dynamics incorporates concepts tested and refined over many centuries, and brought together by Isaac Newton in his famous book whose Latin title translates to *Mathematical Principles of Natural Philosophy*. The book is commonly known as the *Principia*.

The Newtonian view of dynamics holds that objects in the universe "naturally" move in straight lines at constant (or perhaps zero) speeds. According to this view, it makes no sense to ask *why* an object moves this way; only if it *doesn't* do we need to provide a reason. The generic name for a cause of such a non-natural motion is *force*. From the *Principia*:

> *Law 1: Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*

How would you write a simple vector equation that restates Newton's condition of "natural" motion? [ ] How would you express "non-natural" motion? [ ]

Newton expresses his second law this way:

> *Law 2: The alteration of motion is ever proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.*

To Newton, the "quantity of motion" or simply "motion" means what we now call *momentum* ($\vec{p} = m\vec{v}$), and Newton's "alteration" is what we would now call "rate of change." If we choose the right units to measure force, we can make proportionality constant equal to 1. Restate Newton's second law in terms of "rate of change of momentum", and show how this leads to the more familiar version $\Sigma\vec{F} = m\vec{a}$.

[ ]

What assumption about mass do you have to make above?

In order for the constant of proportionality to be 1, what are the SI units of force? [ ]

This SI unit is named the *Newton*.

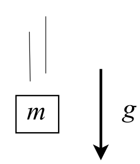
Newton's third law, often called the law of "action and reaction," states that forces *always occur in pairs*, and that each force of a pair acts on a *different object*:

> *Law 3: To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

These pairs of forces are called *third law pairs*.

Weight

We saw in the lab that, neglecting air resistance, an object in free-fall accelerates at about $9.8\ \text{m/s}^2$, which we call g. We name the force causing this acceleration *weight*. Applying Newton's second law to this situation, we express the weight W of an object as mg.



Since forces always occur in pairs, what is the other force that makes a third law pair with the weight of this object?

[ ]

Identifying Forces

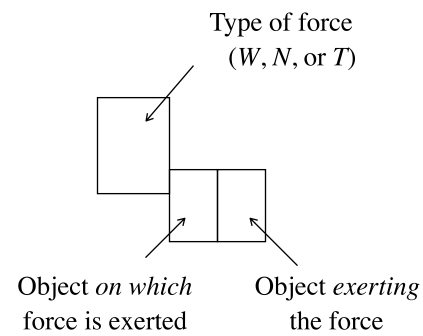
For the time being, we consider only three types of forces (we'll add a few more later):

- *Weight (W)* – the non-contact force on any object on or near the surface of the earth, due to the gravitational force between the earth and the object. Any object near the surface of the earth has weight *mg*.
- *Normal Force (N)* – the pushing force due to contact between two surfaces. It is exerted perpendicular to the surfaces.
- *Tension (T)* – the pulling force due to a string or chain. It is exerted along the string from the end points toward the middle.

When we write the symbol for a force, we include three pieces of information:

1. the *type* of force (from the list above),
2. the object *on which* the force is acting, and
3. the object *exerting* the force.

The last two are written as subscripts for the type of force. For example, consider a book a rest on a table. We identify the weight of the book as: $\vec{W}_{\mathrm{BE}}$, the *weight* exerted *on* the book *by* the earth.



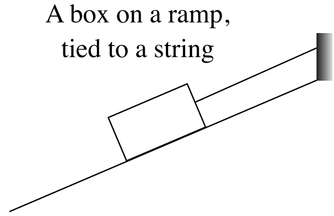
Notation for Forces

Drawing Free-Body Diagrams

A free-body diagram is an abstract representation of an object showing all the forces acting *on* it. We draw a small dot or box to represent the object, and vectors with their tails on the object to show the individual forces (from the list above). We label the forces using the symbols above. The *net* force is never drawn as such in the free-body diagram; it is the vector sum of the individual forces.

Draw a free-body diagram for the box on a ramp, shown to the right, which involves one of each kind of force.

A box on a ramp, tied to a string

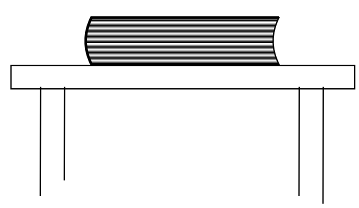


Free-body diagram for box

The vectors drawn on a free-body diagram are not necessarily to scale. We calculate their relative magnitudes based on Newton's laws.

Draw a free-body diagram for the book resting on a table, shown to the right.

A book resting on a table



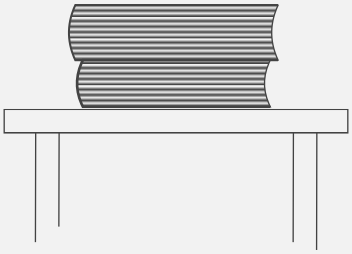
Free-body diagram for book

How do you know that there must be more than one force acting on the book?

What observation allows you to find the relative magnitudes of the forces in the free-body diagram of the book?

A student makes the following statement: "The two forces in the free-body diagram are the same size and point in opposite directions, so they must be a third law pair." Explain why this statement is incorrect, and identify the correct third law pair for each of the forces.

Two books resting on a table



Free-body diagram for upper book

Free-body diagram for lower book

A second book of greater mass is placed on top of the first book as shown above. Draw free-body diagrams for each book in the spaces above.

A student identifies the force exerted on the lower book by the upper book as a weight force. Why is this incorrect?

Rank the magnitudes of all the forces in the two free-body diagrams, and list all third-law pairs.

Describe how the notation on page 3.2 allows us to easily identify third-law pairs.

Applications of Newton's Laws

Applying Newton's laws involves the following steps:

1. Identify the "system" as opposed to the "environment." The *system* is *what is accelerating*. The *environment* is *what is causing the system to accelerate*. Very often the solution of a problem involves choosing more than one system/environment pair. In any case, all of the parts of the system *must share the same acceleration*.
2. Draw a free body diagram showing all forces acting on the system, using *on* and *by* subscripts. Forces between parts of the system (not between the system and the environment) always occur in pairs and will cancel out. They are called internal forces, and are usually not drawn.
3. Choose x and y axes, and positive and negative directions. Choose axes so that one of the axes points in the direction of the known acceleration (eg. down an inclined plane). When circular motion is involved, choose centripetal and tangential axes, and choose as the positive direction toward the center of the circle. Resolve all forces along the chosen x and y axes.
4. Apply $\Sigma\vec{F} = m\vec{a}$ along each of the chosen axes separately. Fill in the left side of the equation ($\Sigma\vec{F}$) by listing the resolved forces for that axis with appropriate signs (+ or –). The *on* and *by* subscripts can be dropped if there is no ambiguity. On the right side, m is the mass of the system you have chosen, and a is its acceleration along the given axis. Solve the resulting equations (sometimes multiple equations in multiple unknowns).

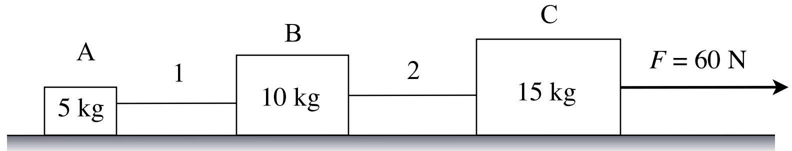
For example, if the x-axis is chosen in the direction of the acceleration, and the y-axis is perpendicular to that, then the last step would be:

$$\underset{\left(\begin{array}{c}\text{list vectors in } x \text{ direction}\\ \text{with +/– signs}\end{array}\right)}{\sum F_x} = \underset{\left(\begin{array}{c}\text{mass of}\\ \text{system}\end{array}\right)}{(m)}\underset{\left(\begin{array}{c}\text{acceleration of}\\ \text{system in } x \text{ direction}\end{array}\right)}{(a_x)} \quad \text{and} \quad \underset{\left(\begin{array}{c}\text{list vectors in } y \text{ direction}\\ \text{with +/– signs}\end{array}\right)}{\sum F_y} = \underset{(0)}{ma_y}$$

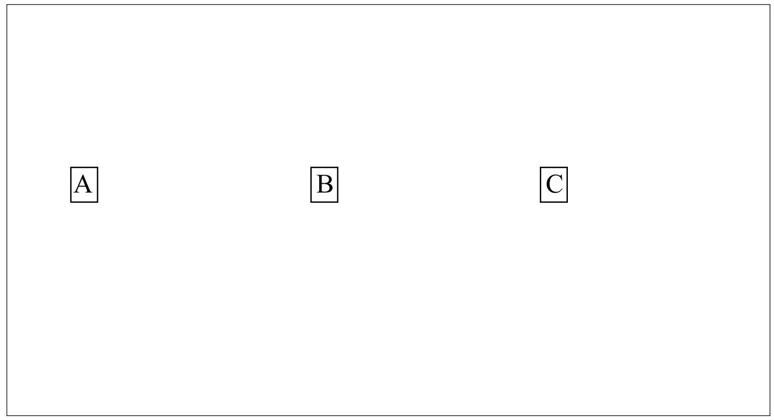
In short:

1. Choose system
2. Draw free body diagram
3. Choose axes and resolve
4. Apply $\Sigma\vec{F} = m\vec{a}$

1. Three blocks A, B and C with the given masses are connected by two strings (1 and 2) on a frictionless surface as shown to the right. Block C is pulled to the right with a force of 60 N.



   A) Draw free-body diagrams of each block in the space provided.



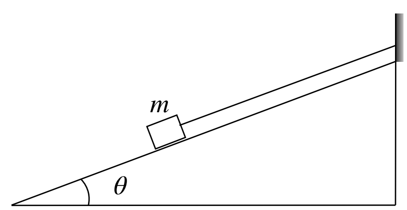
   B) Find the acceleration of the system of blocks. Indicate your choice of system.

   C) Find the tension in string 1. Indicate your choice of system.

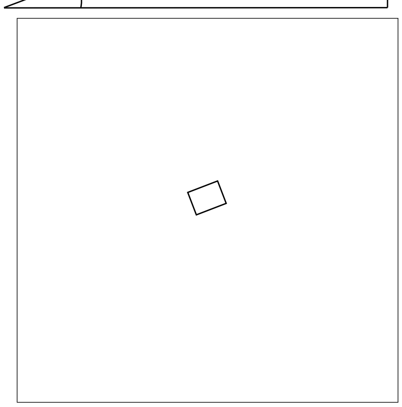
   D) Find the tension in string 2. Indicate your choice of system.

   E) Find the tension in string 2 by choosing a different system from the one in part D).

2. A block of mass m rests on a frictionless inclined plane making an angle θ with the horizontal as shown to the right. A string, tied to the wall at the top of the plane, keeps the block from sliding down the incline.



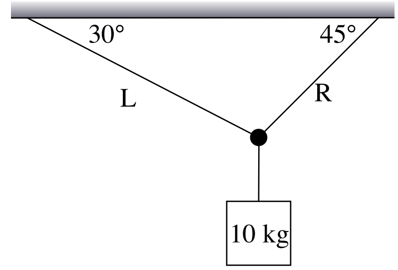
   A) Draw a free-body diagram of the block.



   B) In terms of m, θ and g, find the tension in the string.

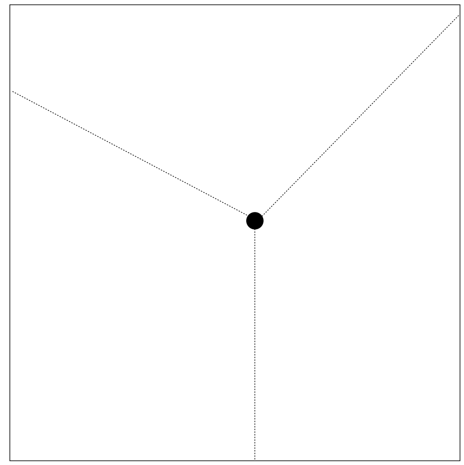
   C) Find the acceleration of the block if the string were cut.

3. A 10 kg block is suspended by a vertical string which is tied to two strings at a knot as shown to the right. The left and right strings make the given angles with the horizontal ceiling.



A) Is the tension in the left string *greater than*, *less than*, or *equal to* the tension in the right string? Or is it impossible to know without a calculation? Explain.

B) In the space to the right, draw and label a vector representing the force on the knot due to the right string. The three dashed lines are for reference, indicating the directions of the strings at the knot.

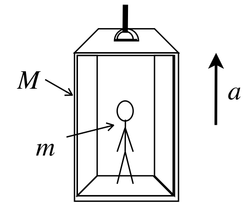


C) In the space to the right, draw and label a vector representing the force on the knot due to the left string. *Draw this vector to a scale consistent with the first vector* and explain how you can determine the scale.

D) In the space to the right, draw and label a vector representing the force on the knot due to the vertical string. *Draw this vector to a scale consistent with the first two vectors* and explain how you can determine the scale.

E) Find the tensions in all three strings.

4. A passenger with a mass m stands on the floor of an elevator of mass M as it accelerates upward with acceleration a.



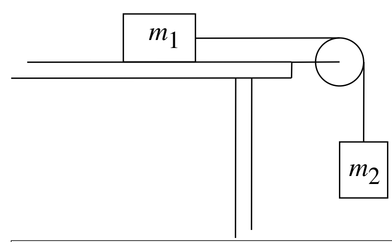
   A) In the space to the right, draw free-body diagrams of the passenger, the elevator, and the entire system.
   B) List all third-law pairs for forces you have drawn.

m M $M + m$

   A) In terms of m, M, a, and g, find the tension in the elevator cable.

   B) Find the normal force of the elevator floor against the passenger.

5. A mass m_1 is on a horizontal frictionless table. A string is connected to m_1 and runs over a massless pulley down to a mass m_2.



   A) Draw and label free-body diagrams of the two masses.
   B) In terms of m_1, m_2, and g, find the acceleration of the system

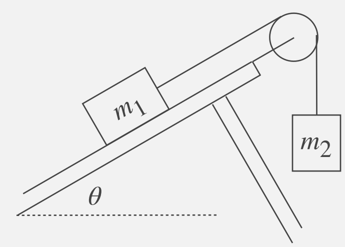
m_1 m_2

   C) Check your results for dimensional consistency and plausibility in the two extreme cases: $m_1 \ll m_2$ and $m_1 \gg m_2$

D) In terms of m_1, m_2, and g, find the tension in the string

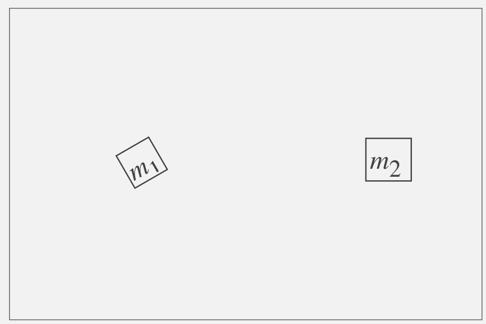
E) Check your results for dimensional consistency and plausibility in the two extreme cases: $m_1 \ll m_2$ and $m_1 \gg m_2$

6. The table from the previous problem is now tilted at an angle θ to the horizontal.



A) Draw and label free-body diagrams of the two masses.

B) In terms of m_1, m_2, θ, and g, find the acceleration of the system.



C) In terms of m_1, m_2, θ, and g, find the tension in the string.

D) In terms of m_1 and m_2, find the angle at which the system would be in equilibrium

Friction

Experimentally we find that when surfaces are in contact there is a force parallel to the surfaces that tends to oppose their *relative* motion. We find that this force is different depending on whether or not the surfaces are actually in relative motion. If they are, we call it the force of *kinetic* friction; if not, we call it *static* friction. We symbolize the frictional force as f, so we now have four types of force for free-body diagrams.

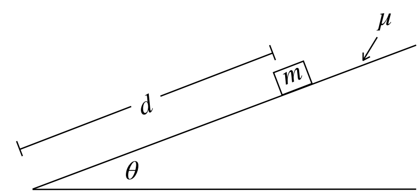
In the case of kinetic friction, the force f_k is found to be proportional to the normal force exerted by each surface on the other. The proportionality constant is called the *coefficient of friction*, written μ_k, so that $f_k = \mu_k N$.

In the case of static friction, the *maximum* frictional force is proportional to N, so that $f_s \le \mu_s N$, but the *actual* frictional force depends on the situation; in general, $f_s = \mu_s N$ if the surfaces are *about to slip*, otherwise it's less.

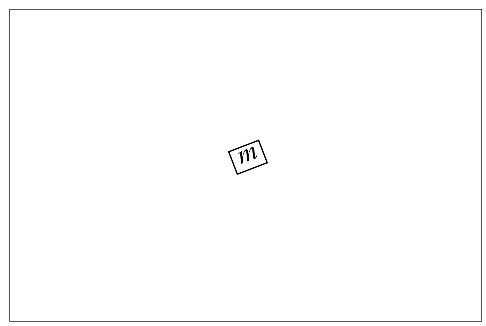
For simplicity, we often ignore the distinction between static and kinetic friction, and just use $f = \mu N$. Although $\vec{f}$ and $\vec{N}$ are vectors, this is not a vector equation because the vectors are always perpendicular to each other.

Examples

1. A block of mass m slides down a distance d from rest on a plane inclined at an angle θ. The coefficient of kinetic friction between the surfaces is μ.



A) Draw and label a free-body diagram of the block as it slides.
B) Find the acceleration of the block.

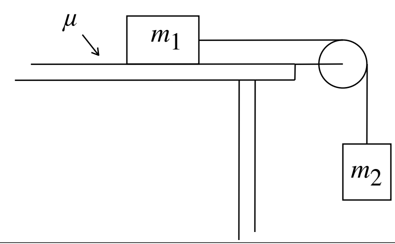


C) Determine the final speed of the block.

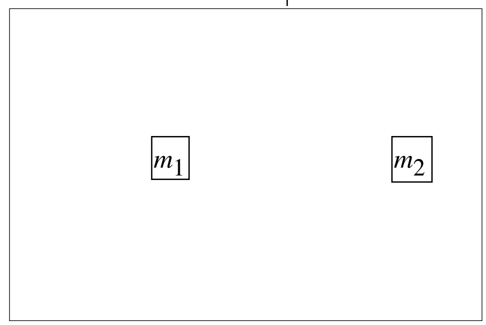
The block continues to slide on a level surface whose coefficient of kinetic friction is also μ.

D) What distance does it go before it stops?

2. A mass m_1 is on a table, where the coefficient of friction between the surfaces is μ. A string is connected to m_1 and runs over a massless pulley down to a mass m_2.



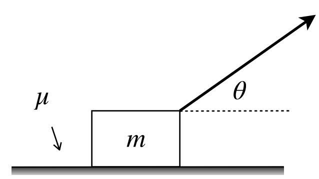
A) Draw and label free-body diagrams of the two masses.
B) In terms of m_1, m_2, μ, and g, find the acceleration of the system



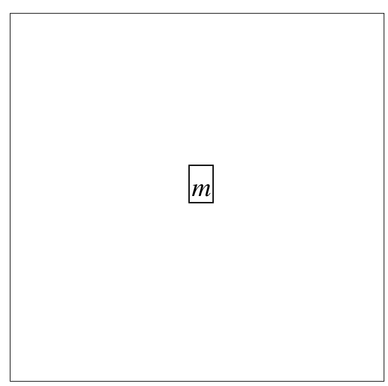
C) In terms of m_1, m_2, μ, and g, find the tension in the string

D) In terms of m_1 and m_2, find the the minimum coefficient of friction necessary to keep the masses from sliding.

3. A block of mass m is pulled across a surface with coefficient of friction μ at constant speed by pulling on a string making an angle θ with the horizontal.



A) Draw and label free-body diagram of the mass.
B) Find the tension in the string.



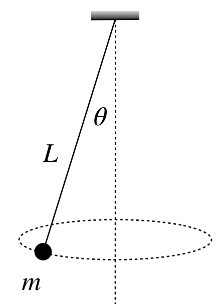
C) At what angle is the tension a minimum?

Circular Motion

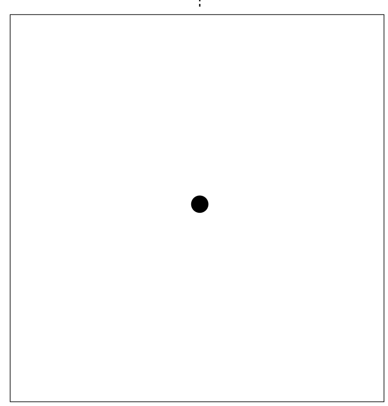
We have seen that when an object travels in a circular path, even for an instant, the component of the acceleration perpendicular to the path has magnitude ____ and points toward the instantaneous center of curvature. If we choose this direction as positive, then Newton's second law for this direction is $\sum F =$ ____. Sometimes circular motion problems are stated in terms of the *period T*, or the time it takes for the particle to cover the circumference of the circle. In this case we can rewrite the Newton's law equation in terms of T as $\sum F =$ ____

Examples

1. A ball of mass m rotates in a horizontal circle at the end of a string of length L at a constant speed. The string makes an angle θ with the vertical, as shown to the right.



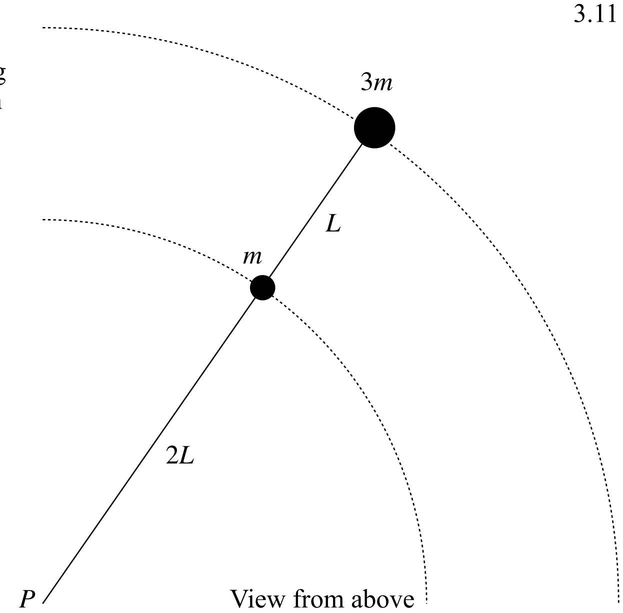
A) Draw a free-body diagram of the ball when it is at the left side of the circle.
B) Find the tension in the string.



C) Find the speed at which the ball rotates.

D) Find the period of rotation of the ball.

2. A small ball of mass $3m$ is connected to another ball of mass m by a string of length L. The second ball is connected by a string of length $2L$ to a pivot P as shown to the right (the view is from above). The assembly is swung in a horizontal circle (we neglect the force of gravity) with period T. In terms of m, T, and L determine:



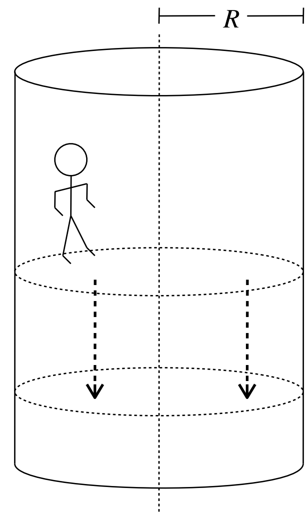
   A) The tension in the string of length L

   B) The speeds of the balls

   C) The tension in the string of length $2L$

   A *single* ball is swung from the string of length $2L$ *only*, with the same period, such that its tension is the same as that found in part C).

   D) What is the mass of the ball?



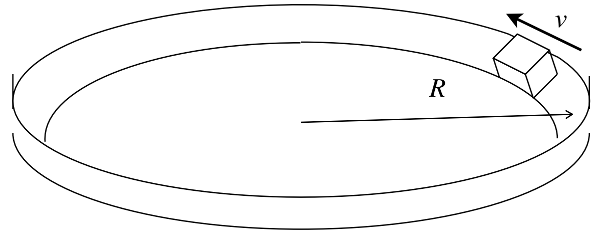
2. A carnival ride consists of a rotating vertical cylinder of radius R with rough canvas walls. The floor is initially about halfway up the cylinder wall as shown to the right. After the rider has entered and the cylinder is rotating sufficiently fast, the floor is dropped down, yet the rider does not slide down. The coefficient of static friction between the walls and the rider is μ, and the rider has a mass m.

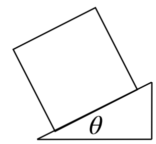
A) Find the minimum speed of the rider at which the floor could be lowered.

B) If the speed of the rider is half the critical speed found in part A), then what is the magnitude of the frictional force in terms of mg?

C) If the speed of the rider is twice the critical speed found in part A), then what is the magnitude of the frictional force in terms of mg?

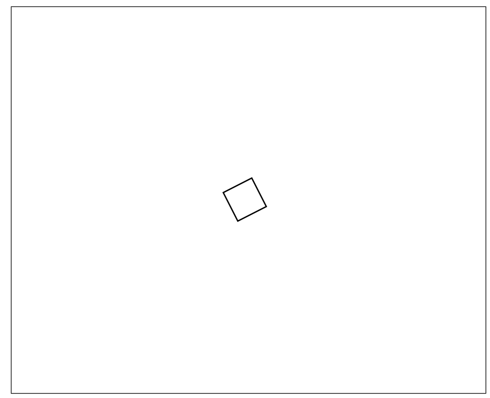
3. A car moves around a frictionless, circular banked track of radius R at a constant speed v. A cross section of the track is shown below, to the right.





A) Draw a free-body diagram of the car below.

B) Find the angle θ at which the track is banked so that the car stays on the track without friction.



The track is now replaced with a track whose inclination angle is θ, but which has a coefficient of friction μ. The car is going faster than the speed v above.

C) Add the appropriate vector to the free-body diagram above (use a dashed line for this vector).

D) Find the maximum speed at which the car could remain on the track.

E) Modify your solution above to find the *minimum* speed at which the car would remain on the track.

Non-constant Forces

Newton's second law gives us a way to find the acceleration of a particle given the various forces on it. Once we find the acceleration we can integrate to find the velocity and position, given the initial conditions.

Forces can depend on various kinematic factors including position, time and velocity. Position-dependent forces will be discussed in Unit 4. Forces that depend on time or velocity cause accelerations that are also time- or velocity-dependent, and we have to be careful not to use the kinematic equations for constant acceleration, but to separate the variables and integrate.

Examples

1. Assume that an object of mass m falling from rest in air experiences force due to air resistance that is proportional to its velocity and directed opposite to its motion. The drag force can be written $F_d = b\,v$, where b is a constant.

   A) Draw a free-body diagram of the object in free fall.

   B) Write $\sum F = m\,a$ for the object.

   C) Write the above equation as a differential equation in v.

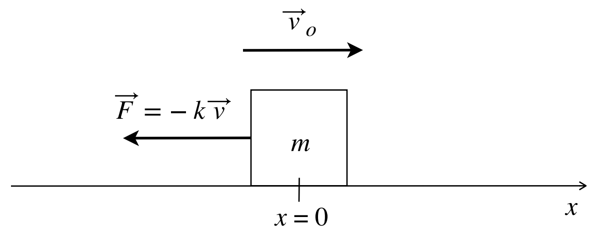
   D) Separate variables and integrate with appropriate limits to find an expression for v as a function of time.

   E) Describe the velocity of the object after a long time.

   F) Write the expression for the acceleration as a function of time.

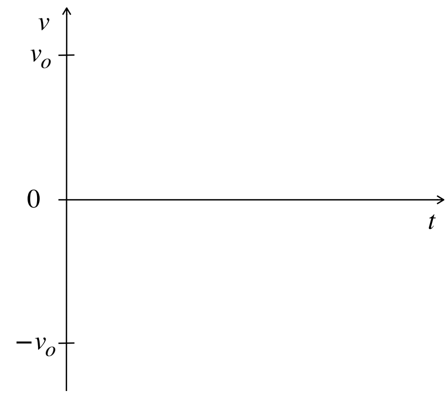
   G) Describe the acceleration initially and after a long time.

2. An object of mass m moving along the x-axis with velocity $\vec{v}$ is slowed by a force $\vec{F} = -k\vec{v}$, where k is a constant. At time $t = 0$, the object has velocity $\vec{v}_o$ at position $x = 0$, as shown to the right.

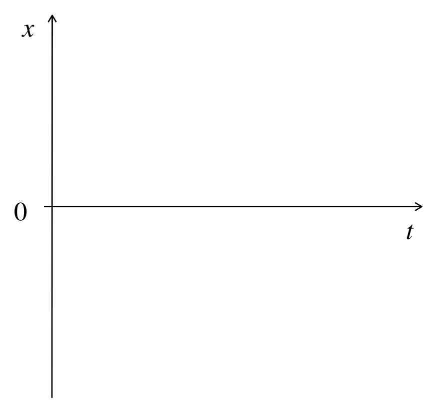


A) What is the initial acceleration (magnitude and direction) produced by the resistant force?

B) Derive an equation for the object's velocity as a function of time t, and sketch this function on the axes to the right. Let a velocity directed to the right be considered positive.

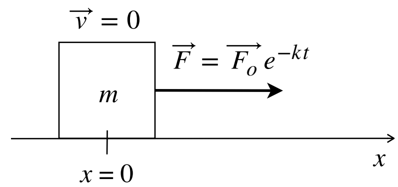


C) Derive an equation for the distance the object travels as a function of time t and sketch this function on the axes to the right.



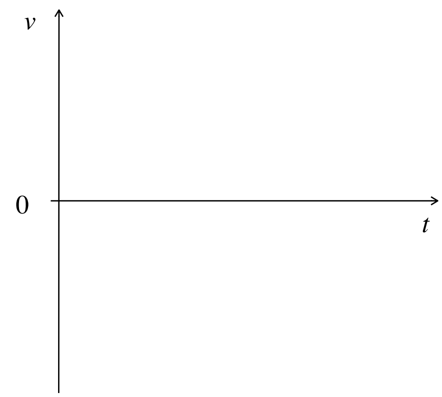
D) Determine the distance the object travels from $t = 0$ to $t = \infty$.

3. An object of mass m is at rest at time $t = 0$, at position $x = 0$, as shown to the right. It is subject to a force $\vec{F} = \vec{F}_o e^{-kt}$, where k is a constant.



A) What is the initial acceleration produced by the force?

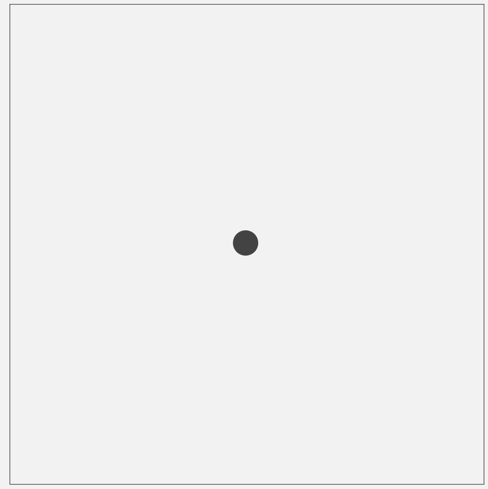
B) Derive an equation for the object's velocity as a function of time t, and sketch this function on the axes below. Indicate any significant points on the v axis.



C) Derive an equation for the object's position as a function of time t.

4. A rubber ball of mass m is dropped from a cliff. As the ball falls, it is subject to air drag (a resistive force caused by the air). The drag force on the ball has magnitude bv^2, where b is a constant drag coefficient and v is the instantaneous speed of the ball. The drag coefficient b is directly proportional to the cross-sectional area of the ball and the density of the air and does not depend on the mass of the ball. As the ball falls, its speed approaches a constant value called the terminal speed.

A) On the figure to the right, draw and label all the forces on the ball at some instant before it reaches terminal speed.



B) State whether the magnitude of the acceleration of the ball of mass m increases, decreases, or remains the same as the ball approaches terminal speed. Explain.

C) Write, but do not solve, a differential equation for the instantaneous speed v of the ball in terms of time t, the given quantities, and fundamental constants.

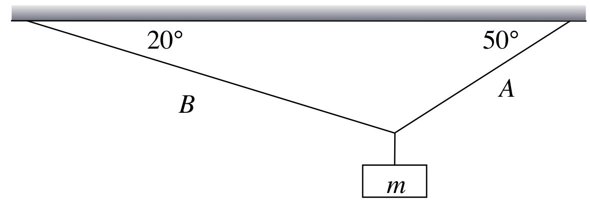
D) Determine the terminal speed v_t in terms of the given quantities and fundamental constants.

AP Physics C
Unit 3 Practice Test

Name ____________________

Part I
Multiple Choice

1. An object of mass m is suspended from three cables as shown to the right. The tension in cable B is



A) greater than mg
B) less than mg
C) equal to mg
D) equal to the tension on cable A
E) there is insufficient information to determine any of the above.

2. In a tug-of-war, two men each pull on the rope with 100 lb. forces, in opposite directions. The tension in the rope is:

A) 100 lb. B) 200 lb. C) zero D) 50 lb. E) 141 lb.

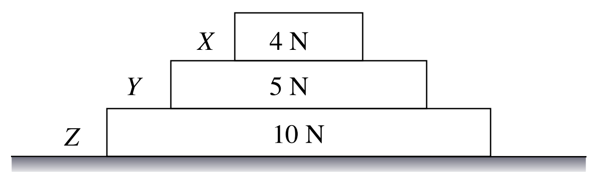
3. A heavy steel ball B is suspended by a cord from a block of wood W. The entire system is dropped through the air. Neglecting air resistance, the tension in the cord is:

A) zero
B) the difference between the masses of B and W
C) the difference between the weights of B and W
D) the weight of B
E) none of these

4. A car moves horizontally with a constant acceleration of 4 m/s^2. A ball is suspended by a string from the ceiling of the car; the ball does not swing, being at rest with respect to the car. What angle does the string make with the vertical?

A) 22°
B) 24°
C) 66°
D) 68°
E) cannot be determined without knowing the mass of the ball

For questions 5 and 6: Three books, X, Y and Z rest on a table. The weight of each book is as indicated to the right.



5. The net force acting on book Y is

A) 4 N down
B) 5 N up
C) 9 N down
D) zero
E) none of these

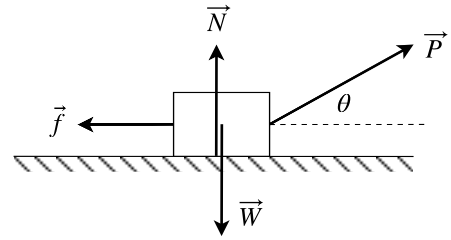
6. The force of book Z on book Y is

A) zero
B) 5 N
C) 9 N
D) 14 N
E) 19 N

7. The coefficient of kinetic friction:

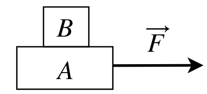
A) is in the direction of the frictional force
B) is in the direction of the normal force
C) is the ratio of force to area
D) can have units of Newtons
E) none of the above

8. A boy pulls a wooden box along a rough horizontal floor at a constant speed by means of a force $\vec{P}$ as shown to the right. Which of the following must be true of the magnitudes of these vectors:



A) $P = f$ and $N = W$
B) $P = f$ and $N > W$
C) $P > f$ and $N < W$
D) $P > f$ and $N = W$
E) none of these

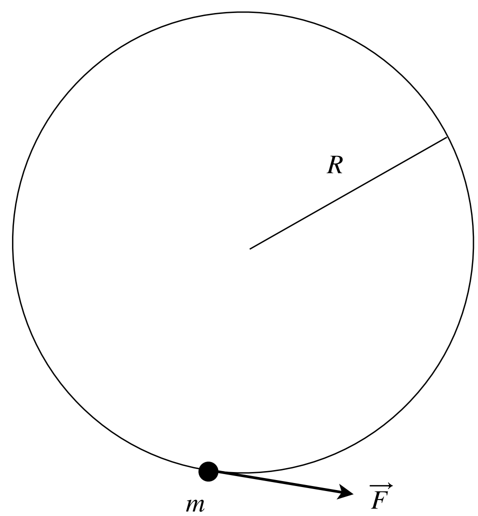
9. Block A, with mass m_A, is initially at rest on a horizontal floor. Block B, with mass m_B, is initially at rest on the horizontal top surface of A. The coefficient of static friction between the two blocks is μ_s. Block A is pulled with an increasing force. It begins to slide out from under B when its acceleration reaches:



A) g
B) $\mu_s g$
C) $m_B \mu_s g$
D) $\dfrac{m_A \mu_s g}{m_B}$
E) $\dfrac{m_B \mu_s g}{m_A}$

10. A bead of mass m slides on a horizontal frictionless circular wire of radius R. It starts from rest at time $t = 0$ and thereafter is pushed by an external force $\vec{F}$ with constant magnitude, always tangent to the wire. The magnitude of the force exerted on the bead by the wire is

A) zero
B) F
C) $\dfrac{Ft}{mR}$
D) $\dfrac{F^2 t^2}{mR}$
E) $\dfrac{F^2}{mg}$

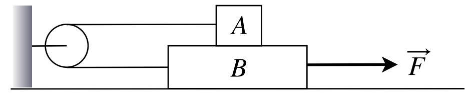


1.B 2.A 3.A 4.A 5.D 6.C 7.E 8.C 9.B 10.D

Part II

Show your work

Credit depends on the quality and clarity of your explanations



1. Block A (mass m_A) rests on block B (mass m_B) and is connected to it by a cord passing around a frictionless pulley as shown to the right. Block B rests on a horizontal surface. The coefficient of kinetic friction between *each pair* of sliding surfaces is μ. A force $\vec{F}$, applied horizontally to block B, maintains the blocks in motion with constant speed.

   A) In the space below, right, draw and label the forces acting on each block. Use subscripts on the force labels to indicate what the force is *on* and what the force is exerted *by*.

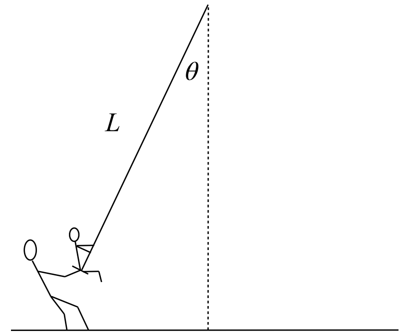
   B) List all pairs of forces that are action-reaction pairs according to Newton's third law.

   C) In terms of m_A, m_B, μ and g, determine:

      i) the tension in the cord

      ii) the magnitude of the applied force F

2. An adult exerts a horizontal force on a swing that is suspended by a rope of length L, holding it at an angle θ with the vertical. The child in the swing has mass m and dimensions that are negligible compared to L. The weights of the rope and of the seat are negligible. In terms of m, g and θ, determine

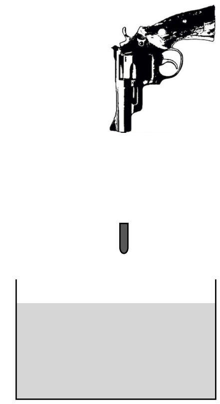


A) the tension in the rope
B) the horizontal force exerted by the adult

The adult releases the swing from rest. As the swing passes through the lowest point, its speed is v.

C) In terms of m, g, L, v and θ, determine
   i) the tension in the rope just after the release (when the swing is instantaneously at rest)
   ii) the tension in the rope as the swing passes through its lowest point
D) For each of the points in parts C) i) and ii), indicate the direction of the net force, and explain your reasoning.

3. A bullet of mass m is shot directly downwards into water. While moving downwards in the water, it experiences a drag force $\vec{F}_d = -k\vec{v}$, where k is a positive constant. The bullet eventually travels at a constant, terminal velocity. In terms of these quantities and constants:



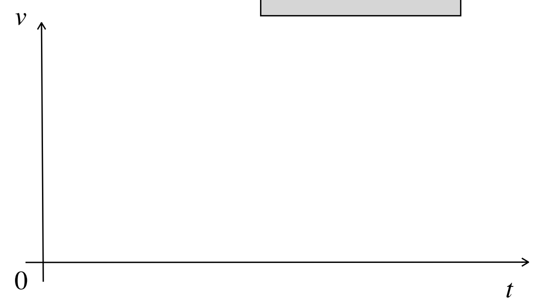
A) Determine the terminal velocity of the bullet while in the water.

The bullet enters the water at a speed v_o which is three times the terminal velocity.

B) Determine the magnitude and direction of the initial acceleration a_o of the bullet as it enters the water.

C) Write the equation for the bullet's velocity v as a function of time from when it enters the water.

D) On the axes below, sketch the graph of the magnitude of the velocity of the bullet as a function of time, labeling significant points on the v axis.

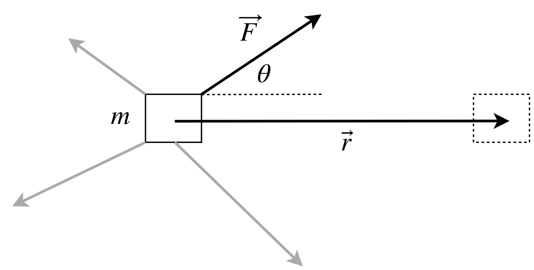


AP Physics C
Unit 4 Name________________________

In Unit 3 we discussed forces that vary with time and velocity. We now begin to consider forces that are functions of position. We'll start with the simplest case, that of a constant force in one dimension, and build up to the general case.

Work

Imagine a mass m that experiences a constant force $\vec{F}$ while moving through a displacement $\vec{r}$. There may be other forces also acting on m, so the directions of $\vec{F}$ and $\vec{r}$ are not necessarily the same. Let the angle between the direction of $\vec{F}$ and the direction of $\vec{r}$ be θ, as in the diagram to the right. We define the *work W* done by the force $\vec{F}$ to be the magnitude of the component of $\vec{F}$ in the direction of $\vec{r}$, multiplied by the magnitude of $\vec{r}$. Since $\vec{F}$ and $\vec{r}$ are both vectors, we have a concise way to express the work done by a constant force:



$W =$

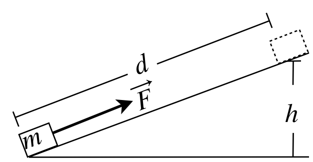
Work is a scalar that can be positive, negative, or zero. Describe the conditions governing the sign of the work W:

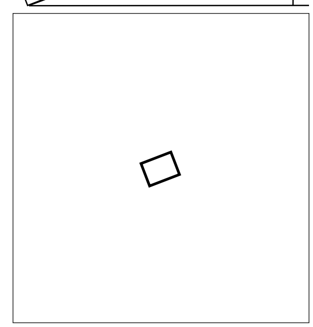
What are the SI units of work? The name for this unit is the *Joule*.

Examples

1. A block of mass m is to be pushed a distance d up an incline so that it is raised to a height h. The surfaces are frictionless, and the block is to be pushed up at a *constant speed*.

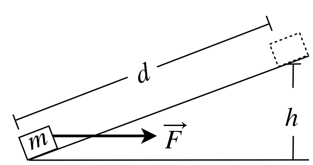
A) If the force is applied parallel to the surface, as in the diagram to the right,
   i) Draw a free-body diagram of the block in the space below.
   ii) Determine the magnitude of the force necessary to move the block up the incline at a constant speed.

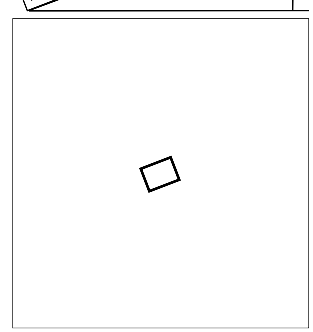




   iii) Find the work done by this force in terms of m, h, and constants.

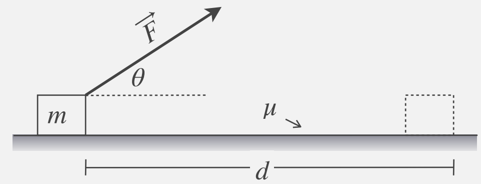
B) If the force is applied horizontally, as in the diagram to the right,
   i) Draw a free-body diagram of the block in the space below.
   ii) Determine the magnitude of the force necessary to move the block up the incline at a constant speed.



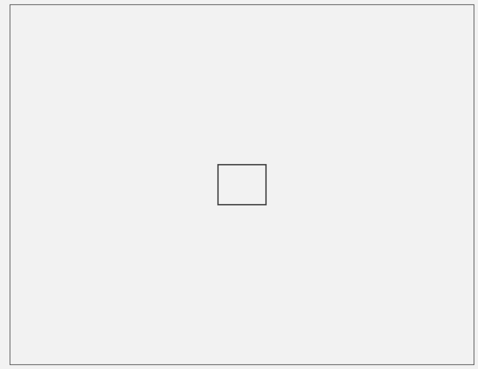


   iii) Find the work done by this force in terms of m, h, and constants.

2. A block of mass m is pulled for a distance d along a horizontal surface at constant speed. The force makes an angle θ with the horizontal as shown to the right. The coefficient of kinetic friction between the surfaces is μ.



A) Draw a free-body diagram of the block in the space below.

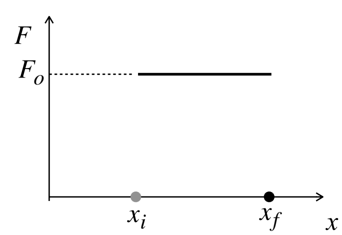


B) Find the magnitude of the force necessary to move the block at constant speed.

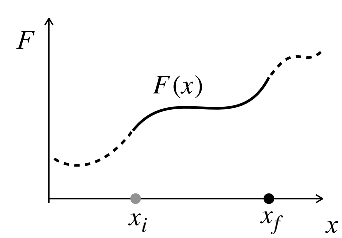
C) Show that the work done by the applied force is given by $\dfrac{\mu m g d}{1+\mu \tan\theta}$.

Work Done By A Variable Force

So far we have been considering the work done by a constant force in the one-dimensional case; that is, a particle that can only move in the $\pm x$ direction, and a constant force F_o that also points only in the $\pm x$ direction. Imagine that the particle moves from x_i to x_f while the force F_o acts. What is the work done by this force? $W =$ The graph to the right of F vs. x represents this situation. How would you represent the work done by F_o on this graph?



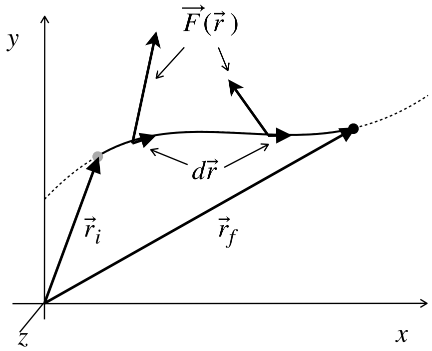
Now consider the case where the force varies as some function of position *F(x)* as shown to the right. Using integral notation, express the work done by this force as the particle moves from x_i to x_f:



Example

The force needed to stretch a spring is proportional to the displacement x of the end of the spring from its equilibrium position, so that $F = kx$ where k is a constant. By integration, find the work needed to stretch the spring a distance x from its equilibrium position (ie., from $x = 0$ to $x = x$).

Variable Force In Multiple Dimensions



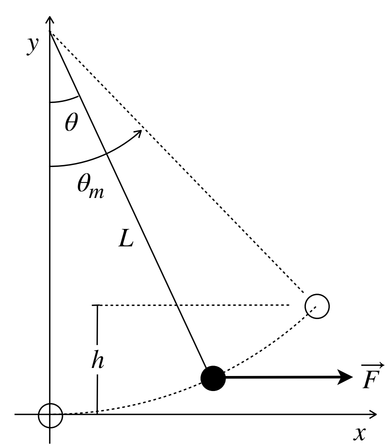
In the general case of a variable force in multiple dimensions, a particle moves on a path from position $\vec{r}_i$ to position $\vec{r}_f$, while a force $\vec{F}(\vec{r}\,)$ varies both in magnitude and direction along the path as shown to the right. For each differential step along the way $d\vec{r}$, the force $\vec{F}$ is practically constant (since $d\vec{r}$ is so short).What is the differential work done during this interval? $dW =$

Use integral notation to express the total work done by the variable force along the entire path from $\vec{r}_i$ to $\vec{r}_f$: $W =$

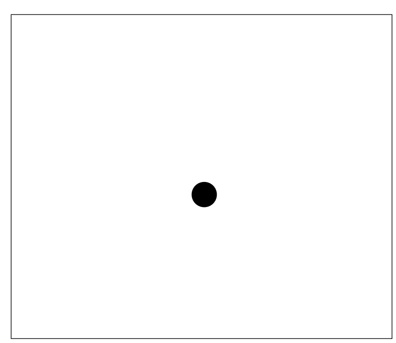
Recall that in three dimensions, the dot product of two vectors, say $\vec{a}$ and $\vec{b}$, can be expressed in terms of their x, y, and z components: $\vec{a} \cdot \vec{b} =$ The expression above for the work done can be written in terms of the components of $\vec{F}$ and $\vec{r}$ as: $W =$

Examples

1. A small object of mass m is suspended from a string of length L. The object is pulled slowly sideways by a force $\vec{F}$ that is *always horizontal*, until the string finally makes an angle θ with the vertical as in the diagram to the right.



A) In the space below, draw a free-body diagram of the object when it is held at an arbitrary angle θ with the vertical.

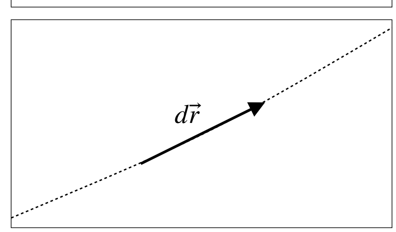


B) Apply Newton's Second Law in the x and y directions at this position.

C) Find the magnitude of the horizontal force as a function of θ.

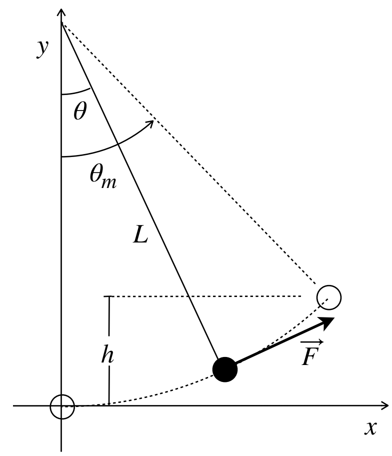
D) Write the component form of the integral for work above. How does this expression simplify in this case? Explain.

E) The diagram to the right represents a close-up view of $d\vec{r}$ when the string is at the angle θ. On the diagram, draw the components $d\vec{x}$ and $d\vec{y}$ of $d\vec{r}$, and identify the angle θ.Write the equation that relates dx, dy and θ.

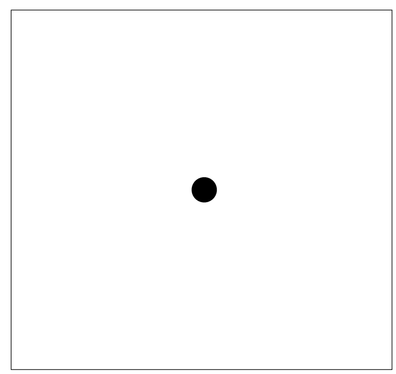


F) Simplify and evaluate the integral from part D), using appropriate limits, to find the total work done by the force $\vec{F}$ in raising the mass to the height h. Express the work done in terms of m, h, and constants.

2. Now the object is pulled slowly sideways by a force $\vec{F}$ that is *always perpendicular to the string*, until the string finally makes an angle θ_m with the vertical and the object is at height h as in the diagram to the right.



A) In the space below, draw a free-body diagram of the object when it is held at an arbitrary angle θ with the vertical by the force perpendicular to the string.



B) Find the magnitude of the force as a function of θ.

C) Write the vector form of the integral for work from the previous page. How does this integral simplify in this case? Explain.

D) Simplify and evaluate the integral from part C), using appropriate limits, to find the total work done by the force $\vec{F}$ in raising the mass to the height h. Express the work done in terms of m, h, and constants.

Conservative Systems and Potential Energy

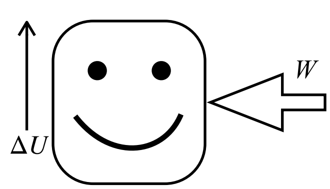
You may have noticed a pattern with the two examples above, and the ones on page 4.1. The work done in raising an object by a certain amount doesn't seem to depend on the path taken to get there, or whether the force is constant or variable. This is true in general of forces that depend only on position, such as the gravitational force in these examples, or the spring force on page 4.2. The work done by one of these forces in moving something from point A to point B is independent of the path taken between these end points. More generally, the work needed to change a system from some initial state to some final state depends only on the initial and final states, not on anything in between.

An equivalent way to describe these forces is that the work done on a round trip from A to B and back to A is zero. Or, the total work is zero if the system returns to its initial state. Such forces are called *conservative forces*, and such systems are called *conservative systems* because they tend to vote Republican.

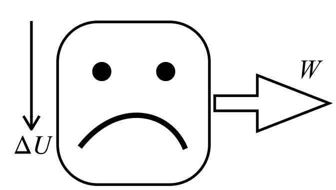
For any conservative system, we define the *potential energy* (symbolized by U) of the current state of the system as the work needed to bring it to its current state from some agreed-upon initial state, whose potential energy is arbitrarily set to zero. The potential energy is thus a relative quantity, but *changes* in potential energy (ΔU) are not; we are usually concerned with changes in energy.

A conservative system gains potential energy when work is done *on* the system, and loses potential energy when work is done *by* the system. If W is the work done *by* the conservative force, then we write:

$$\Delta U = -W$$



Work done *on* system



Work done *by* system

Use the definition of potential energy on the previous page to express the following:

- The gravitational potential energy of an object of mass m at height h above an arbitrary zero level:

  $U_g =$

- The elastic potential energy stored in a spring with spring constant k stretched by a distance x from its relaxed length: $U_e =$

Work Done by the Net Force

Various amounts of work can be done by each of the forces acting on an object. A special case is the work done by the *net* force $\Sigma\vec{F}$, which is in general *not* one of the forces on the object. Newton's Second Law applies only to the *net* force.

Write the vector equation for the work done by the *net* force $\Sigma\vec{F}$: $W_{\Sigma\vec{F}} =$ Use Newton's Second Law to rewrite this in terms of the mass and acceleration: $W_{\Sigma\vec{F}} =$ Use the definitions of $\vec{a}$ and $\vec{v}$ to express the work in terms of $\vec{v}$ and $d\vec{v}$: $W_{\Sigma\vec{F}} =$

Now imagine that the net force changes the velocity of the object from an initial velocity $\vec{v}_o$ to a final velocity $\vec{v}$. Write the above expression as a definite integral with appropriate limits: $W_{\Sigma\vec{F}} =$ Both $\vec{v}_o$ and $\vec{v}$ have x, y, and z components in three dimensions. Using the component version of the dot product, express the work done by the net force in terms of the components of these vectors, with appropriate limits:

$W_{\Sigma\vec{F}} =$

Integrate each term and evaluate at the limits. Rearrange to express the work in terms of the initial and final *speeds* v_o and v:

$W_{\Sigma\vec{F}} =$

We define the *Kinetic Energy K* of an object of mass m moving with speed v as $\frac{1}{2}mv^2$. The equation above is known as the *Work-Energy Theorem.* It says that:

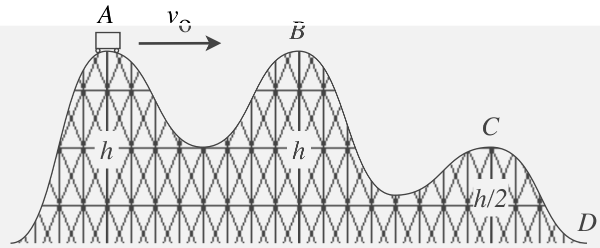
The work done by the net force is equal to

Conservation of Energy

In the early 20th century, German mathematician Emmy Noether proved a beautiful and far-reaching theorem connecting symmetries in nature with conservation laws. The fact that the laws of physics seem to be the same at all times is known as a *symmetry*; she showed that this particular symmetry implies that the *total mechanical energy E* of an isolated system, consisting of the sum of its potential and kinetic energies ($E = U + K$) is conserved. We will see other symmetries and their associated conservation laws later.

Examples

1. A frictionless roller-coaster car starts at point A with speed v_o, as shown to the right. Points A and B are a height h above the ground, point C is at $h/2$, and point D is at ground level.



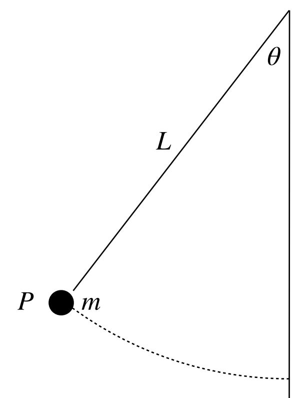
   A) What will the speed of the car be at point B? Explain using the principle of conservation of energy.

   B) What will the speed of the car be at point C?

   C) What will the speed of the car be at point D?

   D) If the car had started from rest at point A (and given a slight nudge) what would the speed of the car be at point D?

2. A ball of mass m is attached to a string of length L and tied to a fixed point, as shown to the right. It is held at point P, so that the string makes an angle θ with the vertical.



   A) If the mass is released from rest, what is its speed as it passes the bottom of its path?

   B) What is the tension in the string at the bottom of its path?

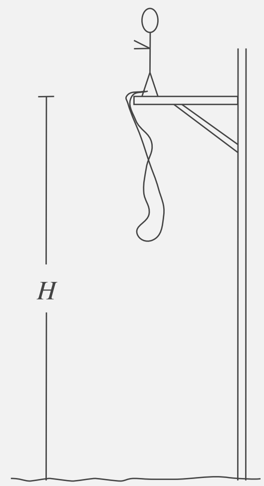
   C) The mass is given an initial speed at point P, and as a result it swings up so that the string is horizontal when it stops. What was its initial speed?

   The mass is given an initial speed at point P, and as a result it swings all the way around in a circle.

   D) If the string is slack as the mass passes the top (ie. the tension is zero), what is the speed of the mass at that point?

E) What is the minimum initial speed at P such that the mass will swing completely around in a circle?

3. A bungee-jumping platform is built a height H from a river. The bungee cord behaves like an ideal spring with stiffness constant k, so that the force it exerts is $F = k\,x$, where x is the distance it is stretched beyond its unstretched length. A jumper of mass m is being prepared for a jump.



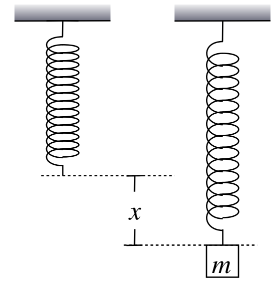
A) What is the length L of cord that should be used so that the jumper stops short of the water? Neglect the height of the jumper in your calculation.

B) What is the tension in the bungee at the instant the jumper stops, before bouncing back up?

C) What is the acceleration of the jumper at the instant he stops?

D) How far above the water is the jumper when his downward speed is a maximum?

4. An ideal spring with a force constant k hangs from the ceiling. We attach a mass m to the spring at its unstretched position and slowly lower the mass until the upward force from the spring balances its weight.



A) Find the distance x that the spring has stretched.

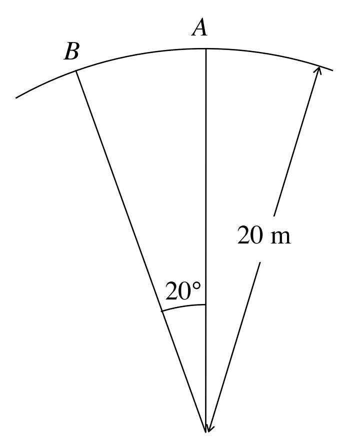
B) Show that the loss of gravitational potential energy of the mass is equal to twice the gain in the spring's potential energy.

C) Why are these two quantities (loss of gravitational potential energy and gain in elastic potential energy) not equal?

Next we raise the mass up to the original unstretched position of the spring and release it from rest there.

D) Determine the speed of the mass as it passes through the point discussed above in terms of k, m, and x.

5. A skier with a mass of 60 kg goes over a frictionless circular hill of radius 20 m. Assume that the effects of air resistance are negligible. As she comes up the hill, her speed is 8.0 m/s at point B.



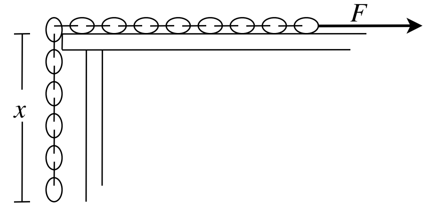
A) What is her speed at the top of the hill (point A) if she coasts over the hill without using her poles?

B) What minimum speed can she have at point B and still coast to the top of the hill?

C) What maximum speed can she have at point B without becoming airborne at point A?

D) Which of the above answers depend on the mass of the skier?

6. A chain of mass m and total length L is held on a frictionless table by a horizontal force F with a portion of length x hanging over the edge, as shown to the right. Assume that the size of the links is small compared to x and L.



A) What is the mass of the portion of the chain hanging over the edge?

B) What is the magnitude of the force F necessary to hold the chain?

A student slowly pulls the chain onto the table by exerting a horizontal force.

C) By integration, find the work done by the student on the chain in pulling it onto the table.

D) By how much has the potential energy of the hanging part of the chain changed? Consider that part as being concentrated at its center of mass (ie. the middle of that part).

E) Explain how this relates to the equation at the bottom of page 4.4, particularly with respect to the sign.

7. A special spring is constructed in which the restoring force is in the opposite direction to the displacement, but it proportional to the cube of the displacement; ie.,$F = -k x^3$. This spring is placed on a horizontal frictionless surface. One end of the spring is fixed, and the other end is fastened to a mass M. The mass is moved so that the spring is stretched a distance A and then released. Determine each of the following in terms of k, A, and M.

A) The potential energy in the spring at the instant the mass is released:

B) The maximum speed of the mass:

C) The displacement of the mass at the point where the potential energy of the spring and the kinetic energy of the mass are equal:

Power

We define *power* as the rate at which work is done: $P = \frac{dW}{dt}$. What are the SI units of power? The name for this unit is the *Watt*. In the British system (used in the US), the unit of power is the *horsepower* (*hp*), originally intended to represent the average power delivered by a workhorse. One hp is equal to 746 Watts.

On page 4.3 you wrote a vector expression for the differential work done by a force $\vec{F}$ moving an object $d\vec{r}$:

$dW =$ Divide both sides of this vector equation by *dt*: Express power in terms of the vectors $\vec{F}$ and $\vec{v}$:

Examples

1. The motor that lifts an elevator is rated at 6 kW. The elevator and its load together weigh 10,000 N, and the floors in the building are 3 m apart. Inside the elevator, a bell rings as it passes each floor. How often does the bell ring?

2. Charles Lindbergh flew across the Atlantic Ocean in 1927 in the *Spirit of St. Louis*, which had a 230 hp motor. If the cruising speed at full power was 50 m/s, what was the force of air resistance on the plane?

One-Dimensional Conservative Systems

Consider again the equation from the bottom of page 4.4. Rewrite this equation in integral form; that is, so that each side is an indefinite integral: If we consider the one-dimensional case for simplicity, the dot product on the right becomes a scalar product: If these two integrals are equal, then the integrands must be equal as well. Solve the resulting expression for the force:

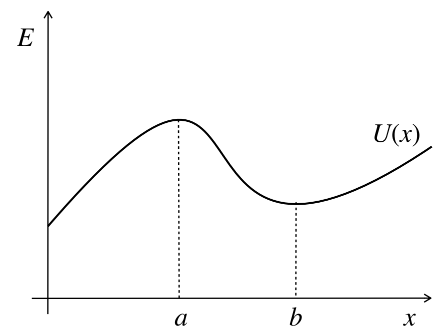
In general, a time derivative is called a *rate* (velocity, acceleration and power are examples). A spatial derivative like the one above is called a *gradient*. Express the equation above that relates force and potential energy in a sentence using this term:

Example

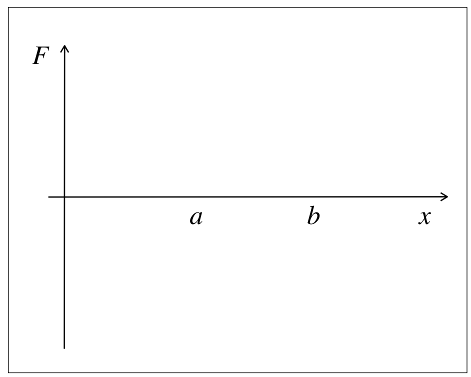
1. The potential energy U of a one-dimensional system is given by $U = a\,x^2 - b\,x$, where a and b are constants,
   A) What are the SI units of a and b?

   B) What is the force function?

2, Consider an arbitrary potential energy function $U(x)$ as shown to the right.
   A) At points a and b, what can you say about the force function? Explain.



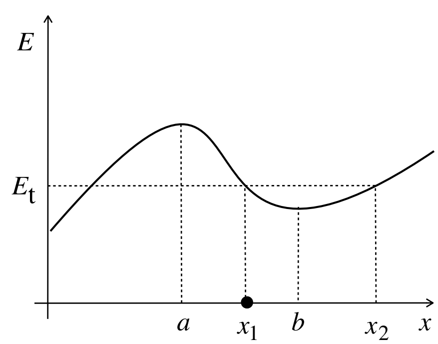
   B) Points a and b are called *equilibrium* points. One is a point of *stable* equilibrium and the other is a point of *unstable* equilibrium. State which is which and explain your reasoning. (Hint: Think about what happens when the object is moved slightly to either side of the equilibrium point.)



   C) To the right make a rough sketch of the $F(x)$ function that corresponds to the $U(x)$ function above

   Now imagine that we have placed a particle at position x_1, between a and b, at rest as shown below. At this point, its total energy is potential, so $E_t = U(x_1)$.

   D) What is the direction of the force felt by this particle? Explain.



   E) The particle moves in response to this force. What is significant about the particle's behavior when it is at point b? Explain.

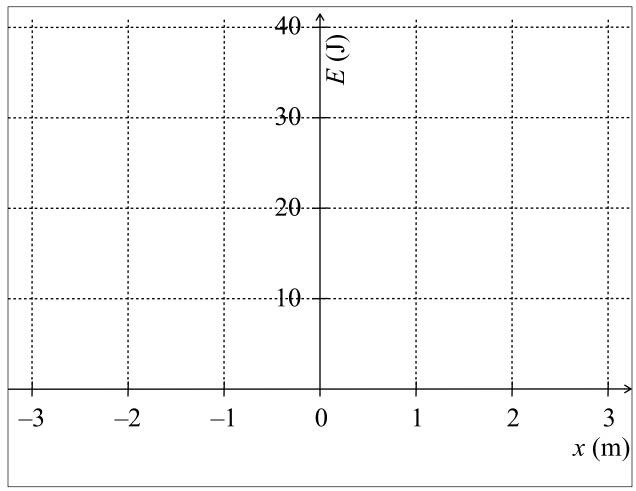
   F) What is significant about the particle's behavior when it is at point x_2? Explain.

   G) Describe the long-term behavior of the particle, and why points x_1 and x_2 are called "turning points."

   H) Will the particle ever reach point a? Explain.

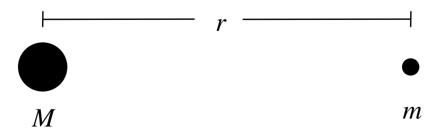
3. Japanese physicist Hideki Yukawa (1907 - 1981) developed a theory of the interaction of nucleons (protons and neutrons) in the nucleus. His theory involved what is called the *Yukawa potential*. The potential energy function is of the form $U(r) = -a\frac{e^{-br}}{r}$, where a and b are constants and r is the distance between the nucleons.
   A) What are the SI units of the constants a and b?
   B) What is the force associated with this potential energy function?

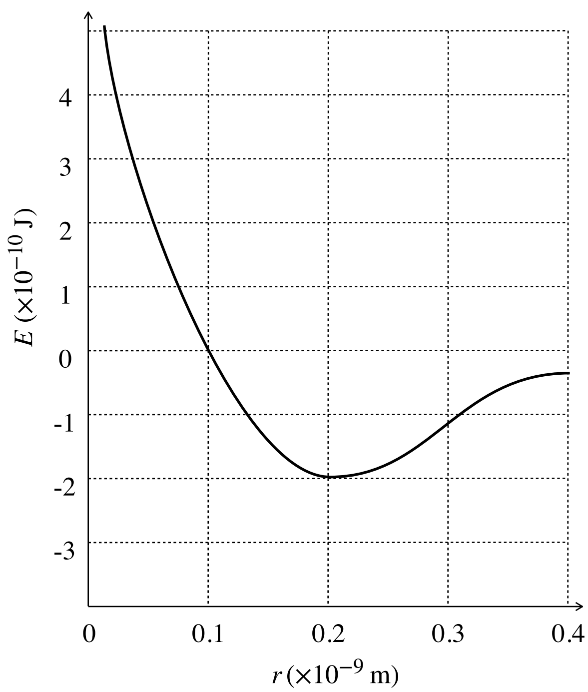
4. The potential energy of a one-dimensional system is given by $U(x) = 4x^2$, where U is in Joules and x is in meters.
   A) What are the SI units of the coefficient 4?
   B) On the axes to the right, sketch a graph of the potential energy function.
   C) An object of mass 0.5 kg is introduced to the system and given a total energy of 20 J. Describe the motion of the object.



   D) Find the speed of the object when it is at $x = 1$ m.
   E) Find the magnitude and direction of the force on the object when it is at $x = 1$ m.
   F) What physical system exhibits this behavior? Explain.

5. The figure below shows a molecule consisting of two atoms m and M (with $m << M$) and separation r. The graph shows the potential energy $U(r)$ of the molecule as a function of r.





A) Describe the motion of the atoms if the total energy of the system is -1×10^{-19} J.

B) Describe the motion of the atoms if the total energy of the system is 1×10^{-19} J.

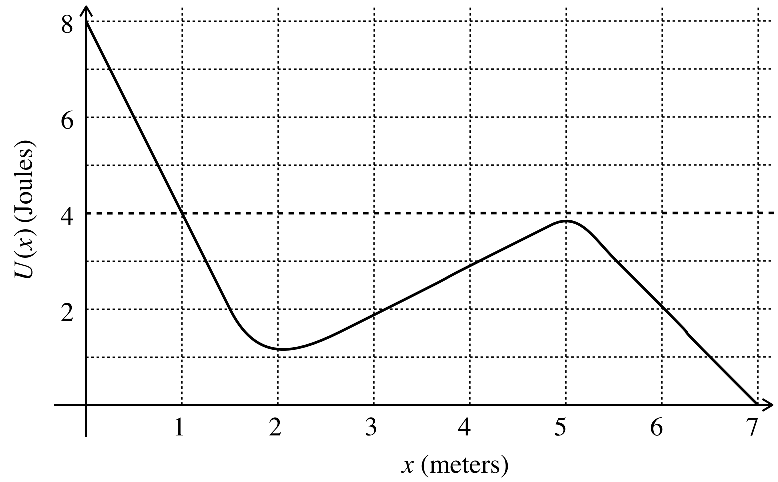
If the total energy of the system is 1×10^{-19} J and $r = 0.3 \times 10^{-9}$ m, use the graph to estimate

C) the potential energy of the molecule.

D) the total kinetic energy of the molecule.

E) the force acting on each atom.

6. The graph to the right shows the potential energy $U(x)$ of a particle as a function of its position x.



A) Identify all points of equilibrium for this particle, and classify them as stable or unstable. Explain your choice.

B) Determine the magnitude and direction of the force on the particle at the following positions

i) $x = 1.0$ m

ii) $x = 4.0$ m

Suppose the particle has a constant total energy of 4.0 joules, as shown by the dashed line on the graph.

C) Determine the kinetic energy of the particle at the following positions

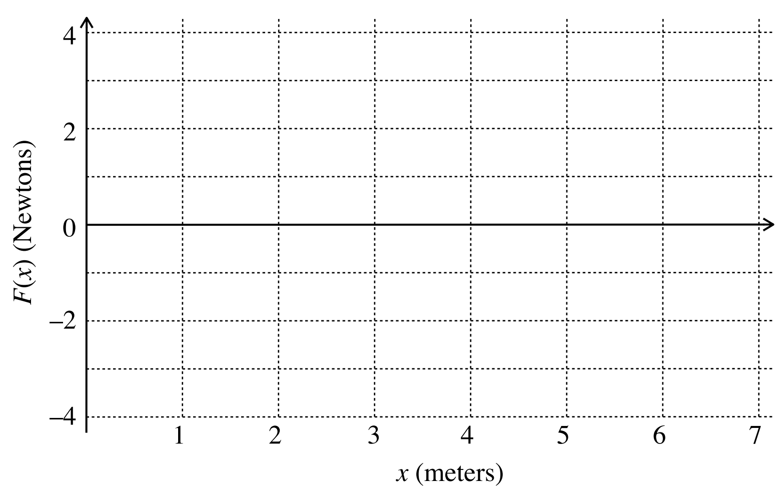
i) $x = 2.0$ m

ii) $x = 4.0$ m

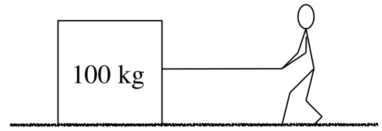
D) Can the particle reach the position $x = 0.5$ m? Explain.

E) Can the particle reach the position $x = 5.0$ m? Explain.

F) On the grid below, carefully draw a graph of the conservative force acting on the particle as a function of x.



7. The 100 kg box shown to the right is being pulled along the x-axis by a student. The box slides across a rough surface, and its position x varies with time t according to the equation $x = \frac{1}{2}\frac{\text{m}}{\text{s}^3}t^3 + 2\frac{\text{m}}{\text{s}}t$.



A) Determine the speed of the box at time $t = 0$.

B) Determine the following as functions of time t.
   i. The kinetic energy of the box

   ii. The net force acting on the box

C) Calculate the net work done on the box in the interval $t = 0$ to $t = 2$ s.

D) State whether the work done on the box *by the student* in the interval $t = 0$ to $t = 2$ s would be greater than, less than, or equal to the answer in part C), and justify your answer.

Two And Three Dimensional Conservative Systems

The equation that relates force to energy in one dimension becomes complicated in two or three dimensions. Potential energy is a scalar function of position, but position is a vector. Force is also a vector. So we want an equation that tells us how to get $\vec{F}(\vec{r}\,)$ from $U(\vec{r}\,)$. The equation is written using the *partial derivative*, which is explained below:

$$\vec{F}(\vec{r}\,) = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

The partial derivative simply means that for the x component of $\vec{F}$ we take the derivative of U with respect to x as if y and z were constants, and similarly for the y and z components. The vector operator *del* (also known as *nabla*), represented by the symbol ∇, is often used to write this more compactly: $\vec{F}(\vec{r}\,) = -\nabla U$,
where $\nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$. By itself, ∇ is not a vector, just like the operator *d/dx* is not a function. But when ∇ acts on a scalar function $U(\vec{r}\,)$, it produces a vector function $\vec{F}(\vec{r}\,)$, just like *d/dx* acting on a function *f* produces a function *df/dx*.

Examples

1. Let $U(\vec{r}\,) = 3x^2y^3z - yz^3 + xz$, where U is in Joules and x, y, and z are in meters.

   A) Find F_x:

   B) Find F_y:

   C) Find F_z:

   Let $\vec{r}_o = \hat{\imath} + \hat{\jmath} + \hat{k}$.

   D) Find $U(\vec{r}_o)$:

   E) Find $\vec{F}(\vec{r}_o)$ expressed in terms of the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$:

2. Consider a particle moving in two dimensions under the influence of a force $\vec{F}$. The potential energy function for the force is $U(x, y) = \frac{1}{2}k(x^2 + y^2)$.

   A) What are the SI units of k?

   B) Find F_x:

   C) Find F_y:

   D) Express $\vec{F}$ in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$:

   E) Express $\vec{F}$ in terms of the position vector $\vec{r}$. Recall that $\vec{r} = x\hat{\imath} + y\hat{\jmath}$:

   F) Describe a physical system that would behave this way.

AP Physics C
Unit 4 Practice Test

Name ____________________

Part I
Multiple Choice

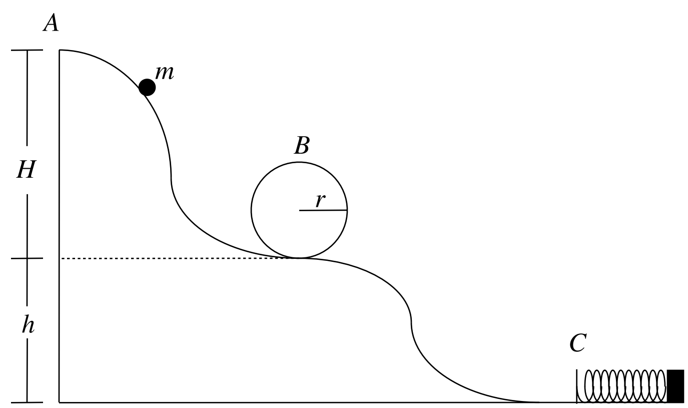
1. A ball of mass m drops with no air resistance. In terms of the distance h that it has fallen, the instantaneous power of the gravitational force is

A) $mg\sqrt{gh/2}$
B) $mg\sqrt{2gh}$
C) $mgh\sqrt{2g}$
D) $2\sqrt{mgh}$
E) $2\sqrt{2gh}$

2. When a certain rubber band is stretched a distance x, it exerts a restoring force of magnitude $F = ax + bx^2$. The work done by an outside agent in changing the stretch from $x = L$ to $x = 2L$ is

A) $(ax + bx^2)L$
B) $a + 2bx$
C) $a + 2bL$
D) $\dfrac{3aL^2}{2} + \dfrac{7bL^3}{3}$
E) $\dfrac{aL^2}{2} + \dfrac{bL^3}{3}$

For questions 3 - 5: A small object of mass m slides down the ramp shown to the right, from rest at point A, around the loop and through point B, and to point C where it strikes a fixed spring of spring constant k. Assume there are no nonconservative forces present.



3. The kinetic energy of the object at point B is

A) 0
B) mgh
C) $mg(h + r)$
D) mgH
E) $mg(H - 2r)$

4. The distance the spring at C is compressed by the object is

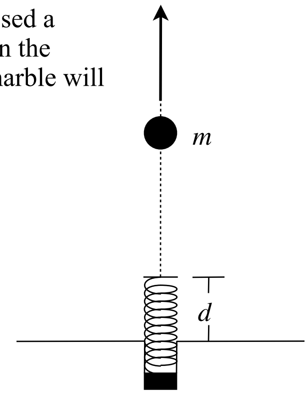
A) $\sqrt{mg/k}$ B) $\sqrt{k/m}$ C) $\sqrt{mgH/k}$ D) $\sqrt{\dfrac{2mg(H+h)}{k}}$ E) $\sqrt{\dfrac{2mg(H+h-r)}{k}}$

5. If the object is not to fall off the ramp as it approaches point B, which of the following conditions must hold?

I The speed of the object at point B must be at least $\sqrt{rg}$
II There must be an upward force whose magnitude is at least mg
III H must be at least $2.5r$

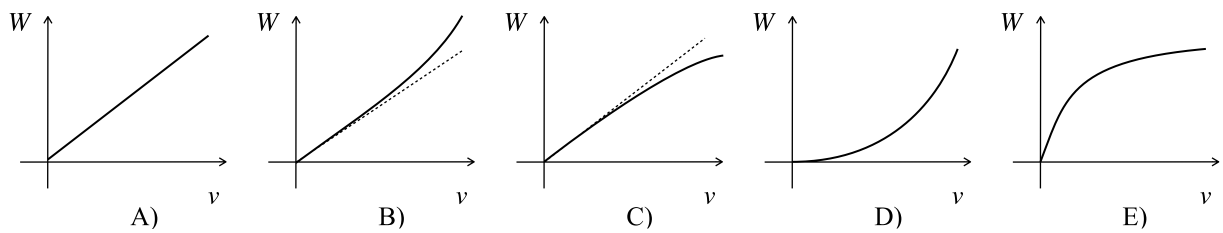
A) I only B) II only C) III only D) I and III only E) II and III only

6. In the diagram to the right, a spring is mounted vertically in a hole so that when it is compressed a distance d from equilibrium the top is flush with the ground. A marble of mass m is placed on the compressed spring, which is then released. Assuming no friction, the maximum height the marble will reach is



A) d
B) $\dfrac{2kd^2}{m}$
C) $\dfrac{kd^2}{2mg}$
D) $\dfrac{2kd^2}{mg}$
E) $\dfrac{kd}{mg}$

7. A particle is initially at rest on a horizontal frictionless table. It is acted upon by a constant horizontal force F. Which of the following graphs best represents the work W as a function of the particle's speed v?



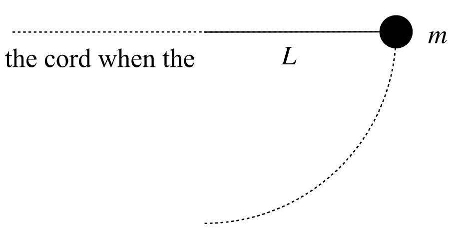
8. As a particle moves along the x axis its kinetic energy increases in proportion to the time t ($K \propto t$). The net force F acting on the particle must be proportional to

A) t^0 (ie. it is constant) B) t C) $1/t$ D) $\sqrt{t}$ E) $\sqrt{1/t}$

9. A block of mass m is initially moving to the right on a horizontal frictionless surface at a speed v. It then compresses an ideal spring of stiffness constant k. At the instant when the kinetic energy of the block is equal to the potential energy of the spring, the spring is compressed a distance of

A) $v\sqrt{\dfrac{m}{2k}}$ B) $\dfrac{mv^2}{2}$ C) $\dfrac{mv^2}{4}$ D) $\dfrac{mv^2}{4k}$ E) $\dfrac{1}{4}\sqrt{\dfrac{mv}{k}}$

10. A small object of mass m, on the end of a light cord of length L, is held horizontally as shown to the right. The object is then released. The tension in the cord when the object is at the lowest point of its swing is

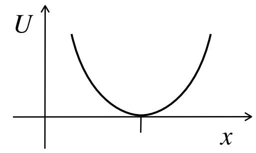


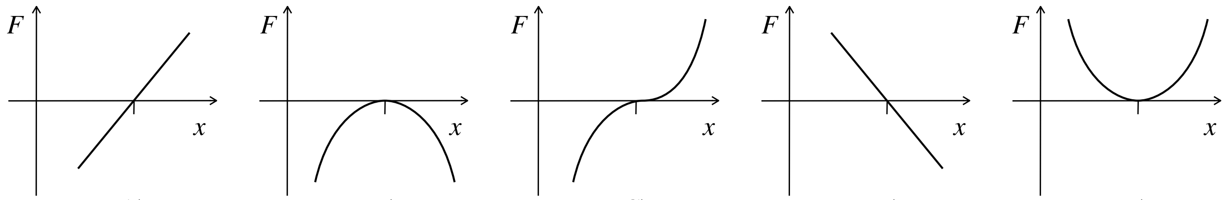
A) $mg/2$
B) mg
C) $2mg$
D) $3mg$
E) mgL

11. Given a potential energy function $U(x)$, the corresponding force $\vec{F}$ is in the positive x direction if

A) U is positive
B) U is an increasing function of x
C) U is negative
D) U is a decreasing function of x
E) It is impossible to obtain the direction of $\vec{F}$ from U

12. The graph to the right shows the a parabolic potential energy function $U(x)$ for a particle moving on the x axis. Which of the following graphs best represents the force F on the particle?

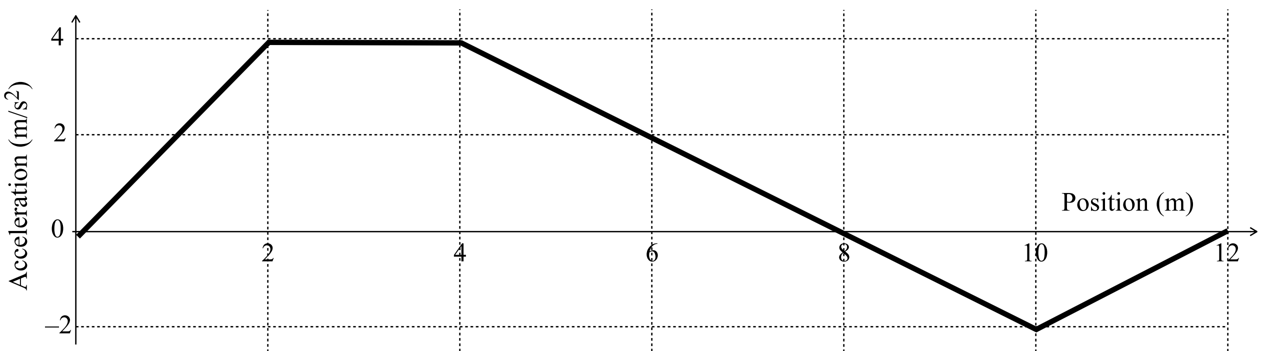




A) B) C) D) E)

13. A ball with mass m projected horizontally off the end of a table with an initial kinetic energy K. At a time t after it leaves the end of the table it has kinetic energy $3K$. Neglecting air resistance the time t is

A) $(3/g)\sqrt{K/m}$
B) $(2/g)\sqrt{K/m}$
C) $(1/g)\sqrt{8K/m}$
D) $(K/g)\sqrt{6/m}$
E) $(2K/g)\sqrt{1/m}$



14. A 2.25 kg mass undergoes an acceleration shown in the graph above. The total work done on the mass is

A) 36 J
B) 22 J
C) 5 J
D) –17 J
E) –36 J

15. A car has an engine which delivers a constant power. It accelerates from rest at time $t = 0$, and at $t = t_o$ its acceleration is a_o. Ignoring energy loss due to friction, its acceleration at $t = 2t_o$ is

A) $\frac{1}{2}a_o$

B) $\frac{1}{\sqrt{2}}a_o$

C) a_o

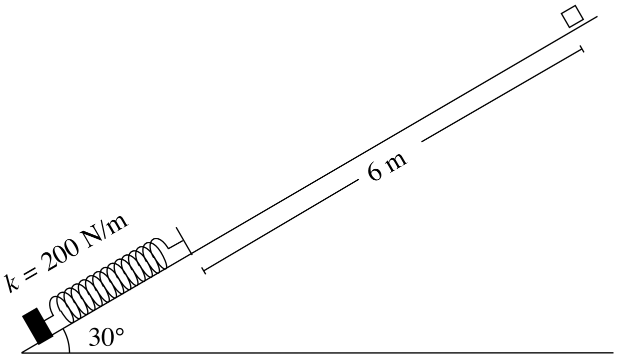
D) $\sqrt{2}a_o$

E) $2a_o$

1. B 2. D 3. E 4. D 5. D 6. C 7. D 8. E 9. A 10. D 11. D 12. D 13. B 14. A 15. B

Part II
Show your work
Credit depends on the quality and clarity of your explanations

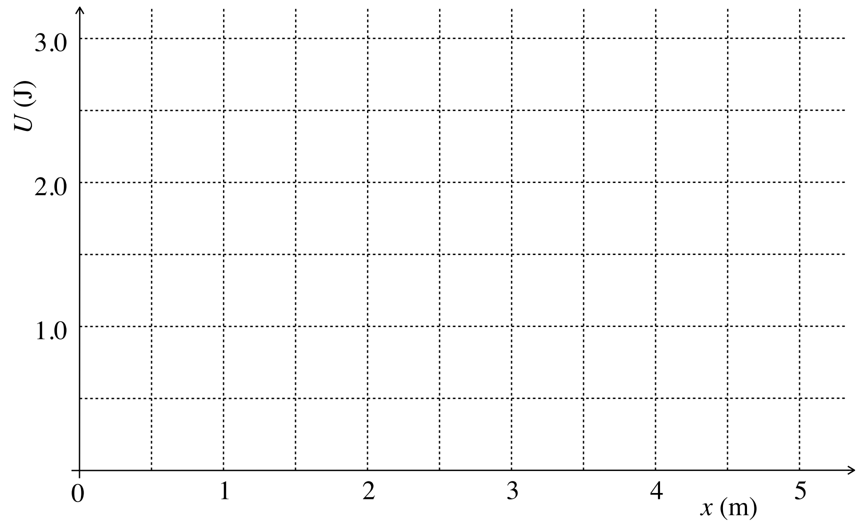
1. A 20 kg mass, released from rest, slides 6 meters down a frictionless plane inclined at an angle of 30° with the horizontal and strikes a spring of spring constant $k = 200$ N/m as shown in the diagram to the right. Assume that the spring is ideal, that the mass of the spring is negligible, and that mechanical energy is conserved. Simplify your calculations by using $g = 10$ m/s^2.



   A) Determine the speed of the block just before it hits the spring.
   B) Determine the distance the spring has been compressed when the block comes to rest. Note that this distance is *not* small compared to 6 meters.
   C) Is the speed of the block a maximum at the instant the block strikes the spring? Justify your answer with appropriate mathematical expressions.

2. An object of mass 0.5 kg experiences a force that is associated with the potential energy function $U(x) = \dfrac{4.0}{2.0 + x}$ where U is in Joules and x is in meters.

   A) Rewrite the potential energy function, assigning the appropriate units to the numerical values so that the equation is dimensionally consistent.
   B) On the axes to the right, sketch a graph of $U(x)$ vs. x.
   C) Determine the force associated with the potential energy function given above. Write the expression for the force using appropriate units as in part A).
   D) Suppose that the object is released from rest at the origin. Determine the speed of the particle at $x = 2$ m.



AP Physics C
Unit 5 Name________________________

So far we have considered the mechanics of individual particles. We now extend our discussion to systems of two or more particles. Rigid, extended objects can be considered systems of particles where the distance between any two particles in the system remains fixed.

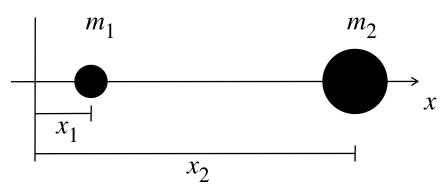
Center of Mass

The ScanTron machine reports that the morning class, with 29 students, had an average of 11.6 on the test, and the afternoon class, with 48 students, averaged 10.1. Show how you would calculate the average score of all students taking the test:

Is it necessarily true that at least one student must have had a score on the test equal to this average? Make up a simple example to support your answer.

The average score on the test is not simply the average of 11.6 and 10.1 (10.85), but a *weighted average*, where each score is assigned a weight based on the number of students in that class.

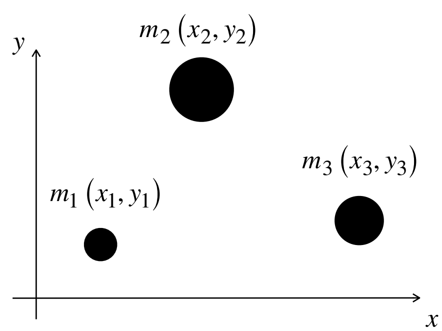
We find that although the motion of a many-particle system can be complex, there is a point (not necessarily coinciding with a particle) that behaves more simply, called the *center of mass*. One way to define the center of mass is the *mass-weighted average position* of the particles in the system.



Consider the simplest case of a many-particle system — one with only two particles. We choose the x-axis such that it passes through their centers as shown to the right. Using the definition above, write a formula in terms of the quantities in the diagram for the x-coordinate of the center of mass of the two particles: $x_{\text{cm}} =$

This definition can easily be extended to systems of three or more particles in one dimension. Use summation notation to write an expression for the x-coordinate of the center of mass of N particles: $x_{\text{cm}} =$

We can also extend this definition to N particles in two or more dimensions. Consider, for example, the system of three particles in two dimensions in the diagram to the right. The center of mass will have both an x- and a y-coordinate; each is determined individually as a mass-weighted average position in that direction. So for this three particle system:



$x_{\text{cm}} =$ $y_{\text{cm}} =$

Now consider a system of N particles in three dimensions. The sum of the masses of the N-particle system is typically abbreviated as simply M. Use both vector and summation notation to express the three-dimensional center of mass position vector: $\vec{r}_{\text{cm}} =$.

In general, the particles that make up the system may each be moving independently. Take the time derivative of both sides of the last equation: Use the concept of weighted average to express this equation in a sentence:

Take the time derivative a second time: Use the concept of weighted average to express this equation in a sentence:

Multiply both sides of the last equation by the total mass M: According to Newton's Second Law, what does each term on the summation side of this equation represent?

Recall that by Newton's Third Law, forces always occur in pairs. The forces acting on each particle of the system may be due to other particles in the system, or they may be external to the system. The summation above can be thought of as: $\Sigma\vec{F}_{\text{ext}} + \Sigma\vec{F}_{\text{int}}$. By Newton's Third Law, what can we say about $\Sigma\vec{F}_{\text{int}}$? Explain.

Considering the system as a whole, then, we can write an equation reflecting Newton's Second Law:

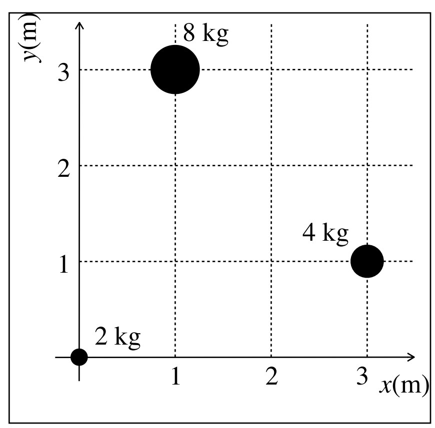
$\Sigma\vec{F}_{\text{ext}} =$ As far as Newton's Second Law is concerned, the system of N particles can be thought of as concentrated at its center of mass.

Example

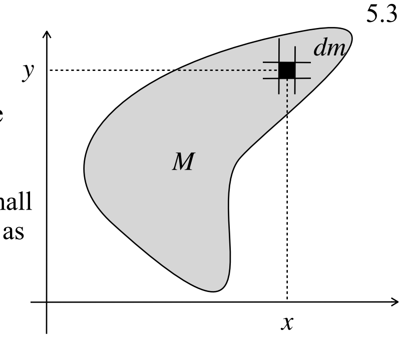
A) Find the x-coordinate of the center of mass of the three objects to the right.

B) Find the y-coordinate of the center of mass of the three objects to the right.

C) Indicate the location of the center of mass on the diagram.



Center of Mass of Extended Objects



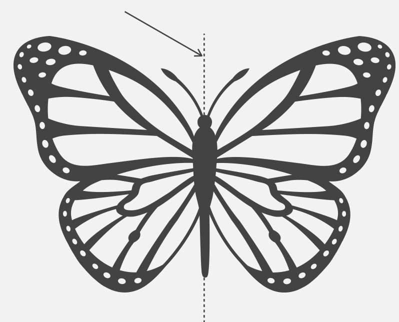
Extended objects can be thought of as being composed of a large number of very small pieces. Instead of adding a finite number of terms in the numerator and denominator, we take an infinite sum of differentially small pieces; that is, we evaluate an integral.

Consider an arbitrarily shaped mass M. Divide it into a large number of differentially small pieces of mass dm, each with its own x- and y-coordinate. Using the summation version as a guide, express the x- and y-coordinates of the center of mass as indefinite integrals:

$x_{\text{cm}} =$ $\qquad$ $y_{\text{cm}} =$

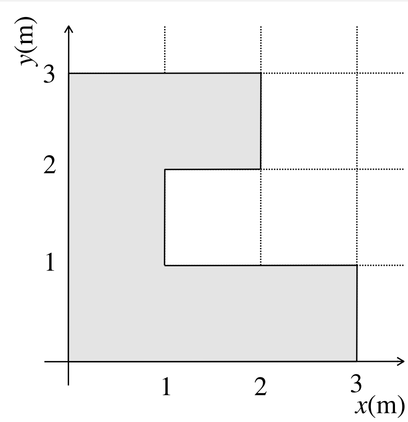
Extend this to three dimensions by expressing the position vector of the center of mass of an arbitrarily shaped three-dimensional mass M as an indefinite integral: $\vec{r}_{\text{cm}} =$

Consider the butterfly drawing to the right. We often use symmetry arguments to simplify the calculation of center of mass of objects. Use the definition of center of mass to explain why the x-coordinate of the center of mass lies on the axis of symmetry. (Hint: Choose a convenient origin for the x-axis.)

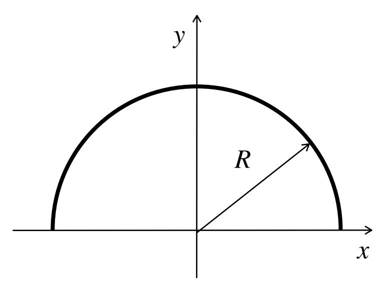


Examples

1. Use symmetry and the definition of center of mass to calculate the x- and y-components of the center of mass of the piece of plywood shown to the right.



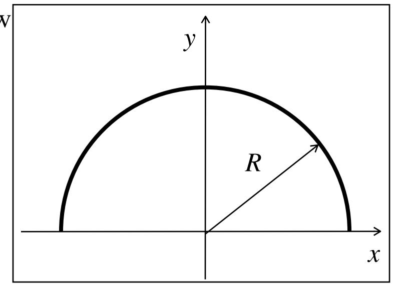
2. Consider a semicircular wire of mass M and radius R, as shown to the right. We wish to find the center of mass of the wire.
   A) Explain why both x_{cm} and z_{cm} are zero.



   B) Predict whether y_{cm} is *greater than*, *less than*, or *equal to* $\frac{R}{2}$. Explain.

C) Choose an arbitrary small piece of the wire and label it dm on the diagram. Draw a line from dm to the origin. Label the angle θ between this line and the x- or y-axis (your choice).

D) The length of dm is a differentially small portion of the arc, say ds. This length subtends a differentially small angle $d\theta$. Label these on the diagram. These geometric parameters uniquely specify the size and location of dm.



E) Write the equation for y_{cm}: $y_{cm} =$

We now use the geometry of the situation to express dm in terms of the geometric parameters.

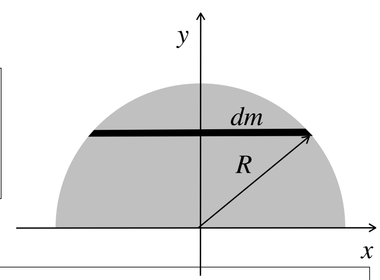
F) Express the ratio $\frac{dm}{M}$ in terms of $d\theta$: $\frac{dm}{M} =$ Express y in terms of θ: $y =$

G) Rewrite the integral above in terms of θ and $d\theta$: $y_{cm} =$

H) Choose appropriate limits and integrate to find y_{cm}.

3. Now consider a thin semicircular plate of radius R and mass M.

A) Explain why both x_{cm} and z_{cm} are zero.



B) Predict whether y_{cm} is *greater than*, *less than*, or *equal to* $\frac{R}{2}$. Explain.

C) Consider the arbitrary small slice of the plate labeled dm in the diagram. Label this slice with parameters that uniquely specify its size and location.

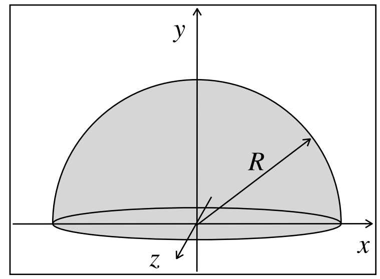
D) Express the ratio $\frac{dm}{M}$ in terms of these parameters: $\frac{dm}{M} =$

E) Write the equation for y_{cm} in terms of one variable:

F) Choose appropriate limits and integrate to find y_{cm}:

4. Now consider a uniform solid hemisphere of radius R and mass M.

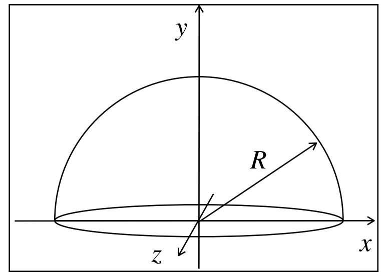
A) Explain why both x_{cm} and z_{cm} are zero.



B) Predict whether y_{cm} is *greater than*, *less than*, or *equal to* $\frac{R}{2}$. Explain.

C) Choose a dm, label it on the diagram with appropriate parameters, find $\frac{dm}{M}$ in terms of these parameters, and integrate to find y_{cm}:

5. Now consider a uniform hemispherical shell of radius R and mass M.

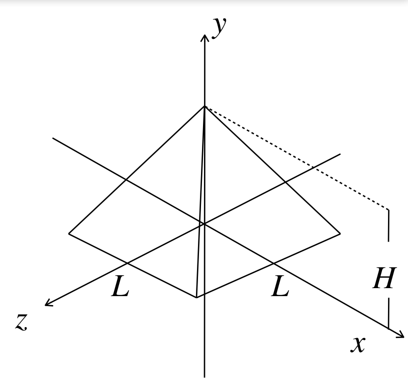


A) Explain why both x_{cm} and z_{cm} are zero.

B) Predict whether y_{cm} is *greater than*, *less than*, or *equal to* $\frac{R}{2}$. Explain.

C) Choose a dm, label it with appropriate parameters, find $\frac{dm}{M}$ in terms of these parameters, and integrate to find y_{cm}:

6. The Great Pyramid of Cheops had a height $H = 147$ m. Its base is a square with edge $L = 230$ m. Its volume V is equal to $\frac{L^2 H}{3}$. Find the height of its center of mass, measured from the ground.



Conservation of Linear Momentum

Recall from Unit 3 that Newton expressed his Second Law in terms of momentum, not acceleration. Take the equation at the top of page 5.2 for $\vec{v}_{cm}$ and rewrite it in terms of momentum ($\vec{p} = m\vec{v}$):

Write Newton's Second Law for a system of N particles in terms of the total momentum $\vec{P}$ of the system: $\Sigma\vec{F}_{ext} =$

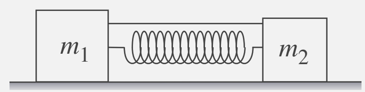
The principle of *conservation of linear momentum* is stated as follows:

In the absence of external forces, the total momentum of an isolated system is conserved.

Explain how this is a consequence of the above equation.

Examples

1. Masses m_1 and m_2 are connected by a compressed ideal spring, and tied together with a light string. The system is at rest on a frictionless surface, when the string is cut and the masses spring apart.

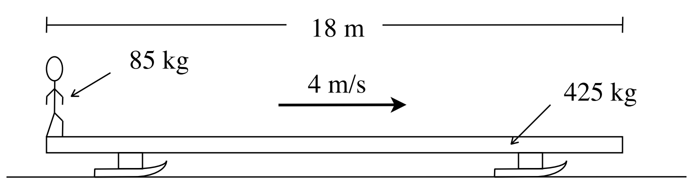


   A) What is the ratio of the velocities of the two masses?

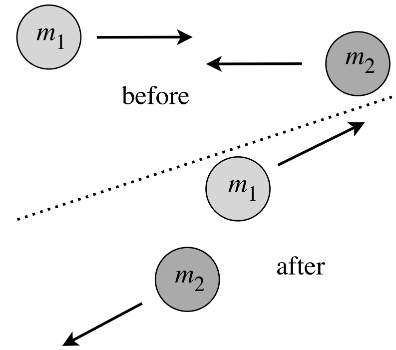
   B) What is the ratio of the kinetic energies of the two masses?

2. Two objects have have equal kinetic energies. If one has three times the mass of the other, which one has the larger momentum, by what factor?

3. An 85 kg man is standing at the rear of a 425 kg iceboat that is moving at 4 m/s across frictionless ice. He decides to walk to the front of the 18 m long boat, and does so at a speed of 2 m/s *with respect to the boat*. Find out how far the man and the boat move with respect to the ice during the time he walks.

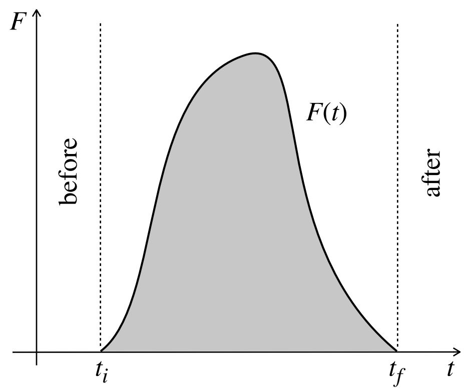


Impulse and Collisions



A very common application of the law of conservation of momentum is that of collisions. During collisions the forces between the objects vary in complex ways that are difficult to account for, but using the conservation of momentum allows us to ignore these details.

By *collision* we mean an event that can be isolated in time, so that there is a clear distinction between *before* and *after*. We only need to consider collisions between pairs of objects, because more complicated collisions can generally be thought of as combinations of pairwise collisions. During the collision the objects exert forces on each other, but by Newton's third law, the forces are equal in magnitude at all times and opposite in direction.



For simplicity we first consider a one-dimensional collision. In the graph we plot the magnitude of the force on one of the objects as a function of time. Write the definite integral representing the shaded area under $\vec{F}(t)$:

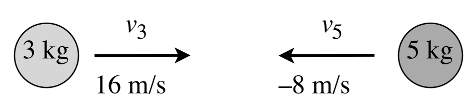
We call this quantity the *impulse*, symbolized by J. Impulse is a vector, whose direction is the same as that of the force. By Newton's Third Law, how does the impulse for one of the objects relate in magnitude and direction to the impulse for the other object?

Assume that the object goes from some initial momentum p_i to some final momentum p_f. Using Newton's statement that force is rate of change of momentum ($F = \dfrac{dp}{dt}$) write the definite integral for impulse in terms of momentum:

$\vec{J} =$

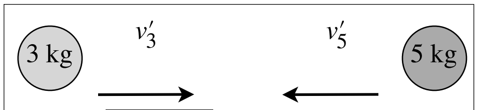
Recall that work (the integral of force with respect to distance) results in a change of energy. Similarly, impulse (the integral of force with respect to time) results in a change of momentum. In a sense, energy and momentum (work and impulse) are related like space and time; in relativity theory they are each united into one four-dimensional entity: *spacetime* and *energy-momentum*. Of the four components of energy-momentum, one is the energy (a scalar) and the other three are the components of the momentum (a vector).

Frames of Reference



The drawing to the right represents two objects on a collision course. Determine the magnitude and direction of the velocity of the center of mass ($\vec{v}_{cm}$) of this system.

It is often convenient to view collisions from other *frames of reference*; that is, coordinate systems that are in motion with respect to the observer. The observer's frame of reference is called the *laboratory frame*. For the system above, consider the frame of reference that is moving with the 5 kg object; that is, at 8 m/s to the left. In this frame of reference, fill in the magnitudes of the velocities v_3' and v_5' of the two objects, and determine the velocity of the center of mass (v_{cm}') of this system in this frame of reference.



Let $\vec{v}_{fr}$ be the velocity of the frame of reference (in this case, –8 m/s). Write an expression to show how to convert from a velocity in the original frame ($\vec{v}$) to a velocity in the new frame ($\vec{v}'$): $\vec{v}' =$ ______

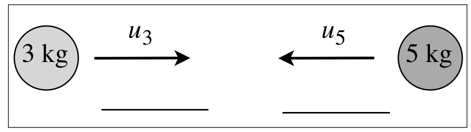
Calculate the momentum of each object, and the total momentum of the system, in the original frame.

3 kg	5 kg	System

Calculate the momentum of each object, and the total momentum of the system, in the new frame.

3 kg	5 kg	System

Now consider a reference frame that moves with the center of mass of the system. We will use the symbol $\vec{u}$ for velocities in this special frame. Fill in the velocities $\vec{u}_3$ and $\vec{u}_5$ of the two objects in the center of mass frame. Calculate the momentum of each object, and the total momentum of the system, in this frame.



3 kg	5 kg	System

Before the collision, the velocity of the center of mass in this frame is clearly zero. What is the velocity of the center of mass of the system *after* the collision in this frame? Explain.

Calculate the kinetic energy of the system in each of the three frames.

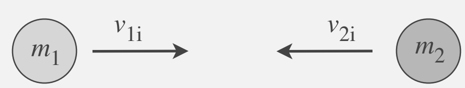
Original frame	Primed frame	cm frame

In each frame, calculate the kinetic energy of an object with the same total mass, moving at the speed of the center of mass.

Original frame	Primed frame	cm frame

The kinetic energy of every system can be thought of as being composed of two parts: $K = K_{cm} + K_{rel}$, where K_{cm} represents the kinetic energy of the particles in the center of mass frame, and K_{rel} represents the kinetic energy of the entire system relative to an external frame. $K = K_{cm}$ only in the cm frame; in all other frames $K > K_{cm}$.

The diagram to the right represents the general case of two objects on a collision course. Derive an expression for the momentum of m_1 in the center of mass frame. Clearly separate the "mass" part and the "velocity" part.



The “mass” part of the expression is called the *reduced mass* of the system, usually symbolized by μ. The “velocity” part is called the *relative velocity of approach*, symbolized by $\vec{v}_{\text{rel i}}$.

Rewrite the expression for the momentum of m_1 in terms of μ and $\vec{v}_{\text{rel i}}$:

After the collision the objects move apart as shown to the right. Write the expression for the momentum of m_1 after the collision, using terms similar to the ones defined above: $\vec{p}\,'_{1f} =$

Kinetic Energy in Collisions

We have seen that in the absence of external forces, the momentum of an isolated system is conserved. For the general case of a one-dimensional collision above, write the equation in terms of the two masses and the initial and final velocities that expresses conservation of momentum:

If the collision happens to be one in which kinetic energy is also conserved, write the equation in terms of these same variables that expresses this:

Eliminate the $\frac{1}{2}$’s in the energy equation and rearrange the terms in each of these equations so that the left side has all the “sub 1” terms and the right side has all the “sub 2” terms.

Divide the energy equation by the momentum equation and rearrange so that all the “sub i” terms are on the left and the “sub f” terms are on the right.

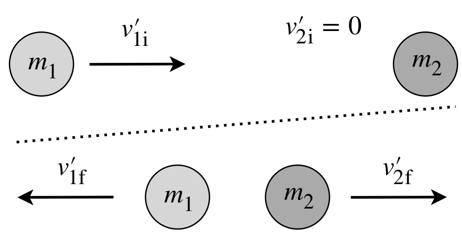
A collision in which the kinetic energy is conserved is called an *elastic collision*. We will define the *elasticity e* of the collision as: $e = \left|\frac{v_{\text{rel f}}}{v_{\text{rel i}}}\right|$. ($e$ is also called the *coefficient of restitution.*) What is the value of e for an elastic collision?

$e =$ Collisions in which e is *less than* this value are called *inelastic*. Collisions in which e is *greater than* this value are called *superelastic*. Collisions in which $e = 0$ are called *completely inelastic*.

In inelastic collisions, kinetic energy is converted into thermal energy. In superelastic collisions, energy is converted from some other form into kinetic energy.

One-Dimensional Elastic Collisions

The diagram to the right represents a one dimensional elastic collision in the reference frame of m_2. Write the simplified equation for conservation of momentum:



Write the simplified equation relating the relative velocities before and after the collision:

Combine these equations to derive equations for the two final velocities in terms of the masses and the initial velocity:

What do your equations predict will happen when $m_1 = m_2$?

What do your equations predict will happen when $m_1 \gg m_2$?

What do your equations predict will happen when $m_1 \ll m_2$?

Rewrite the two equations for the final velocities using velocities in the laboratory reference frame.

Show that these equations are equivalent to the equations to the right.

$$\vec{v}_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{v}_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)\vec{v}_{2i}$$

$$\vec{v}_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)\vec{v}_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)\vec{v}_{2i}$$

Example

Assume that the collision on page 5.8 is elastic.

A) Use the equations above to determine the final velocities of the two objects in the laboratory reference frame.

B) Use the equations on the previous page to determine the final velocities of the two objects in the cm frame.

C) How do the final velocities compare to the initial velocities in the cm frame?

Completely Inelastic Collisions

Now assume that the collision on page 5.8 is completely inelastic. Calculate the final velocities in each frame:

Original frame	Primed frame	cm frame

Calculate the final kinetic energy of the system in each frame:

Original frame	Primed frame	cm frame

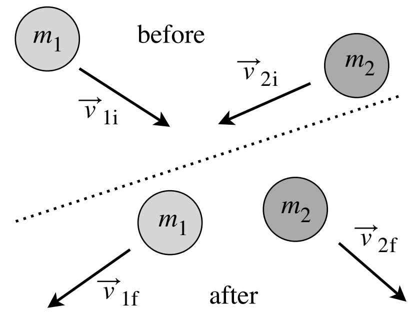
Calculate the loss of kinetic energy in each frame (see p. 5.9 for the initial kinetic energies):

Original frame	Primed frame	cm frame

Make a general statement about the loss of kinetic energy in each frame, as compared to K_{cm} and K_{rel} (page 5.9):

A student makes the following statement: "A completely inelastic collision is one where all the initial kinetic energy is lost to thermal energy." What's wrong with this statement? Make a correct statement similar to the student's.

Collisions in Two Dimensions



When objects collide at an angle (called a *glancing* collision), momentum is conserved in each direction independently. Write the vector form of the conservation of momentum equation for this collision:

As a special case, consider an elastic, glancing collision between objects of equal mass, with one of them initially at rest. This is typical of collisions between the cue ball and a stationary object ball on a pool table.

A) Write the simplified version of the conservation of momentum equation in vector form:

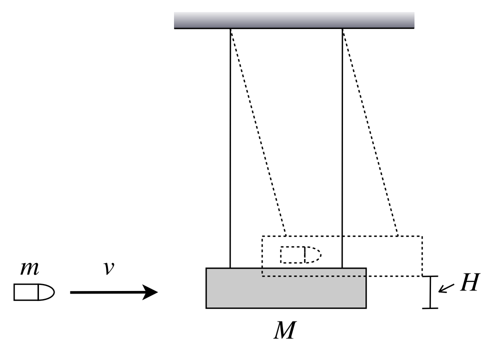
B) Write the simplified version of the conservation of kinetic energy equation:

C) What do these two equations, taken together, say about the angle between $\vec{v}_{1f}$ and $\vec{v}_{2f}$? Explain.

Examples

1. A *ballistic pendulum* is used to measure the velocity of a bullet. It consists of a soft wooden block of mass M suspended from a support. A bullet of mass m is shot into the block, making a completely inelastic collision. The block + bullet swings up to a height H, which is measured, and from these quantities (the two masses and H), the initial velocity of the bullet is calculated.

   A) If the system swings up to the height H, use conservation of energy to find the speed at which it was moving immediately after the collision.



   B) Use conservation of momentum to find the speed of the bullet before the collision.

2. A neutron of mass m makes a head-on elastic collision with a nucleus of mass M, initially at rest. What fraction of the neutron's initial kinetic energy does it transfer to the nucleus in the collision?

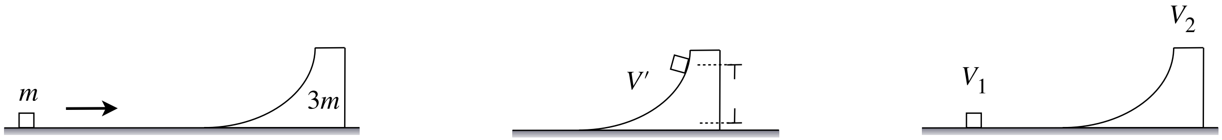


Figure I　　Figure II　　Figure III

3. A block of mass m slides at velocity v_o across a horizontal frictionless surface toward a large curved movable ramp of mass $3m$ as shown in Figure I. The ramp, initially at rest, has a frictionless face up which the block can slide. When the block slides up the ramp, it momentarily reaches a maximum height as shown in Figure II and then slides back down to the horizontal surface as shown in Figure III. In terms of v_o and g:

A) Find the velocity V' of the moving ramp and block at the instant the block reaches its maximum height.

B) Find the maximum height to which the block rises.

C) Determine the final speeds V_1 and V_2 of the block and ramp after the block returns to the level surface. State whether the block is moving to the right or to the left.

4. A swing of mass M is connected to a fixed point by a massless cord of length 3 m. A child also of mass M sits on the seat and begins to swing with zero velocity at a position at which the cord makes a 60° angle with the vertical as shown in Figure I. The swing continues down until the cord is exactly vertical at which time the child jumps off in the horizontal direction. The swing continues in the same direction until the cord makes a 45° angle with the vertical as shown in Figure II.

A) With what velocity relative to the ground did the child leave the swing?

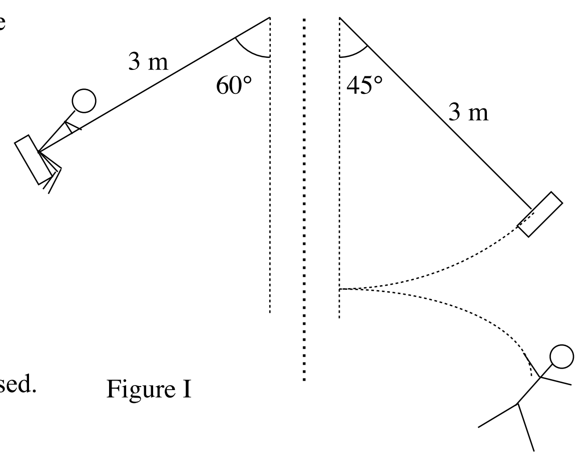
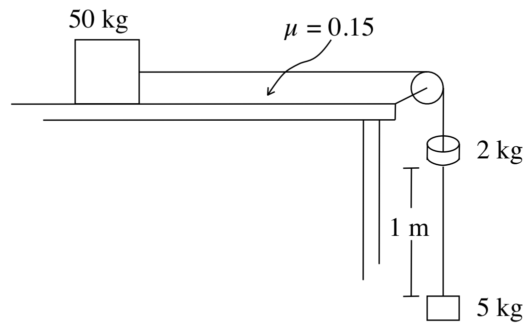


Figure I

B) Show that the total mechanical energy of the system has increased. Where did the additional energy come from?

5. A 50 kg block at rest on a table is connected by a string to a 5 kg block over a frictionless pulley as shown to the right. Assume that the coefficient of kinetic friction between the block and the table is 0.15. At the instant the system is released from rest a ring that has a mass of 2 kg is dropped from a height of 1 meter above the 5 kg weight. Assume that the ring comes to rest on the weight without bouncing.



A) Determine the speed of the 2 kg ring just before it hits the 5 kg block.

B) Determine the speed of the system immediately after the collision.

C) Determine the total distance that the 50 kg block moves

D) Calculate the total potential energy lost by the system by the time it comes to rest.

E) Calculate the thermal energy generated by friction on the table.

F) Explain why your answers to D) and E) are not equal.

G) Calculate the missing energy directly and compare your result to the difference between your answers to D) and E)

6. A chain of mass m and length L is held with its lowest link just touching a table as shown in Figure I to the right. The chain is dropped from that position and coils up in a small heap on the table. Assume that each link is small compared to L, and that it stops immediately upon hitting the table. Figure II shows the chain as it is falling, and a short section of length dy, which was originally at a height y, is now hitting the table, coming to rest in a short time dt.

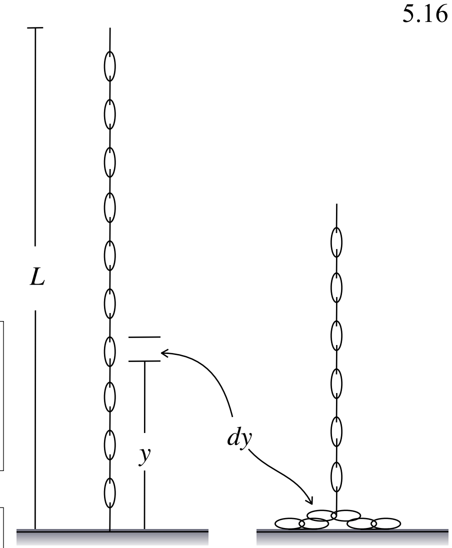


Figure I Figure II

A) At the instant depicted in Figure II, what is the normal force exerted by the table on the part of the chain that has already dropped?

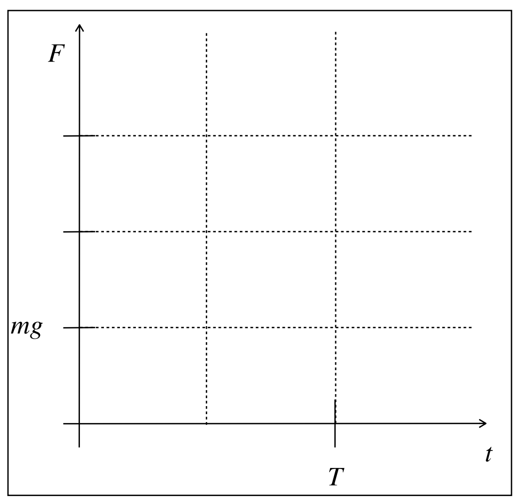
B) What is the speed of the section dy just before it hits the table?

C) What is the mass of the section dy, in terms of m and L?

D) What is the change in momentum of the section dy during the time dt?

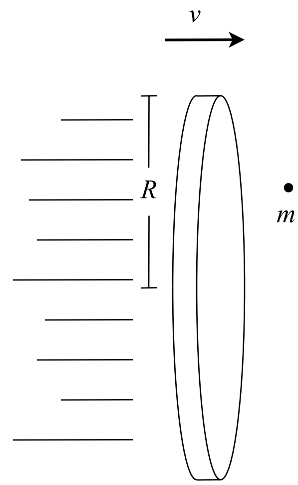
E) What is the rate of change of momentum of the section dy during the time dt?

F) What is the total normal force exerted by the table at the instant depicted in Figure II?



G) In terms of mg, the weight of the chain, sketch a graph of the normal force exerted by the table from $t = 0$, when it was dropped, until after it has fallen to the table. Let $t = T$ be the time when the last link hits the table.

7. **Air Resistance** Consider a disk of radius R and mass M moving at a speed v through air as shown to the right. Imagine air to consist of small molecules of mass m ($<< M$) at rest.



A) From the discussion on page 5.11, what would be the result of an elastic collision between the disk and the molecule?

B) What would be the change in momentum of the molecule in such a collision?

C) If there are N such molecules per unit volume in air, how many molecules would the disk encounter in a time dt?

D) Calculate the total (differential) momentum transferred to the air in the time dt.

E) Find the force exerted by the disk on the air, which is equal and opposite to the force exerted by the air on the disk.

F) Let the mass density of air be ρ. How is ρ related to m and N?

G) Express the force of air resistance as a function of ρ, R and v.

AP Physics C
Unit 5 Practice Test

Name ____________________

Part I
Multiple Choice

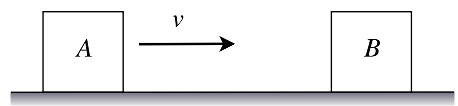
1. If two objects have the same momentum, but different masses, which of these is possible?

   A) The one with less mass has more kinetic energy.
   B) The one with more mass requires more impulse to bring it to that momentum from rest.
   C) The same work is done to each to accelerate it from rest.
   D) They each have the same velocity.
   E) None of the above.

2. A force $F = At$ acts on an object from time $t = 0$ to time $t = T$, where A is a constant. The impulse given to the object over this time period is

   A) A
   B) AT
   C) $AT/2$
   D) $AT^2/2$
   E) AT^2

For questions 3 - 5: Object A moves at a speed v and collides with object B, initially at rest as shown to the right. Both objects have the same mass. Assume the surface on which the objects move to be frictionless.



3. If the collision is completely inelastic, the speed of object A after the collision is

   A) 0 B) $v/4$ C) $v/2$ D) v E) $2v$

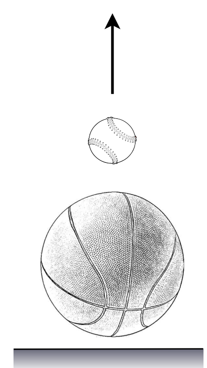
4. If the collision is elastic, the speed of object A after the collision is

   A) 0 B) $v/4$ C) $v/2$ D) v E) $2v$

5. If the collision is elastic, the speed of object B after the collision is

   A) 0 B) $v/2$ C) v D) $2v$ E) $4v$

6. A baseball is dropped on top of a basketball. The basketball hits the ground, rebounds with a speed of 4.0 m/s, and collides with the baseball as it is moving downward at 4.0 m/s. After the collision, the baseball moves upward as shown in the figure and the basketball is instantaneously at rest right after the collision. The mass of the baseball is 0.2 kg and the mass of the basketball is 0.5 kg. Ignore air resistance and ignore any changes in velocities due to gravity during the very short collision times. The speed of the baseball right after colliding with the upward moving basketball is



   A) 4.0 m/s
   B) 6.0 m/s
   C) 8.0 m/s
   D) 12.0 m/s
   E) 16.0 m/s

7. Two rockets are in space in a negligible gravitational field. All observations are made by an observer in a reference frame in which both rockets are initially at rest. The masses of the rockets are m and $9m$. A constant force F acts on the rocket of mass m for a distance d. As a result, the rocket acquires a momentum p. If the same constant force F acts on the rocket of mass $9m$ for the same distance d, how much momentum does the rocket of mass $9m$ acquire?

   A) $p/9$
   B) $p/3$
   C) p
   D) $3p$
   E) $9p$

8. A ball of mass m_1 travels along the x-axis in the positive direction with an initial speed of v_0. It collides with a ball of mass m_2 that is originally at rest. After the collision, the ball of mass m_1 has velocity $v_{1x}\,\hat{i} + v_{1y}\,\hat{j}$ and the ball of mass m_2 has velocity $v_{2x}\,\hat{i} + v_{2y}\,\hat{j}$. Consider the following five statements:

   I) $m_1 v_{1x} + m_1 v_{2x} = 0$

   II) $m_1 v_0 = m_1 v_{1y} + m_2 v_{2y}$

   III) $m_1 v_{1y} + m_2 v_{2y} = 0$

   IV) $m_1 v_0 = m_1 v_{1x} + m_1 v_{1y}$

   V) $m_1 v_0 = m_1 v_{1x} + m_2 v_{2x}$

   Of these five statements, the system must satisfy

   A) I and II B) III and V C) II and V D) III and IV E) I and III

9. A bullet of mass m_1 strikes a pendulum of mass m_2 suspended from a pivot by a string of length L with a horizontal velocity v_0. The collision is perfectly inelastic and the bullet sticks to the pendulum bob. The minimum velocity v_0 such that the bob (with the bullet inside) completes a circular vertical loop is

   A) $2\sqrt{Lg}$

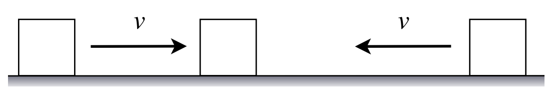
   B) $\sqrt{5Lg}$

   C) $\dfrac{(m_1+m_2)\,2\sqrt{Lg}}{m_1}$

   D) $\dfrac{(m_1-m_2)\sqrt{Lg}}{m_2}$

   E) $\dfrac{(m_1+m_2)\sqrt{5Lg}}{m_1}$

For questions 10 - 11: Three blocks of identical mass are placed on a frictionless table as shown. The center block is at rest, whereas the other two blocks are moving directly towards it at identical speeds v. The center block is initially closer to the left block than the right one. All motion takes place along a single horizontal line.



10. Suppose that all collisions are instantaneous and perfectly elastic. After a long time, which of the following is true?

   A) The center block is moving to the left.
   B) The center block is moving to the right.
   C) The center block is at rest somewhere to the left of its initial position.
   D) The center block is at rest at its initial position.
   E) The center block is at rest somewhere to the right of its initial position.

11. Suppose, instead, that all collisions are instantaneous and perfectly inelastic. After a long time, which of the following is true?

   A) The center block is moving to the left.
   B) The center block is moving to the right.
   C) The center block is at rest somewhere to the left of its initial position.
   D) The center block is at rest at its initial position.
   E) The center block is at rest somewhere to the right of its initial position.

12. Consider a completely inelastic collision between two lumps of space goo. Lump 1 has mass m and originally moves directly north with a speed v_0. Lump 2 has mass $3m$ and originally moves directly east with speed $\frac{1}{2}v_0$. What is the final speed of the masses after the collision? Ignore gravity, and assume the two lumps stick together after the collision.

    A) $\frac{7}{16}v_0$

    B) $\frac{\sqrt{5}}{8}v_0$

    C) $\frac{\sqrt{13}}{8}v_0$

    D) $\frac{5}{8}v_0$

    E) $\sqrt{\frac{13}{8}}v_0$

13. Two equal mass objects moving at the same speed collide head-on and rebound with speeds equal to twice their initial speeds. Which of the following is true?

    I) Since momentum wasn't conserved, there must have been external forces acting.
    II) Energy stored in the objects must have been released.
    III) This is an example of an elastic collision.

    A) I only B) II only C) III only D) I and II only E) I and III only

14. A cart full of sand is rolling along a frictionless surface as a hole in the bottom of the cart allows sand to fall out. As the cart rolls and the sand falls out the speed of the cart will

    A) increase at a constant rate.
    B) increase at a non-constant rate.
    C) decrease at a constant rate.
    D) decrease at a non-constant rate.
    E) remain the same.

15. A railroad car of mass m is moving with velocity v when it collides with a second railroad car of mass 2m at rest. The two cars lock together instantaneously and move along the track. The kinetic energy converted into thermal energy in the collision is

    A) $\frac{mv^2}{9}$

    B) $\frac{mv^2}{4}$

    C) $\frac{mv^2}{3}$

    D) $\frac{mv^2}{2}$

    E) mv^2

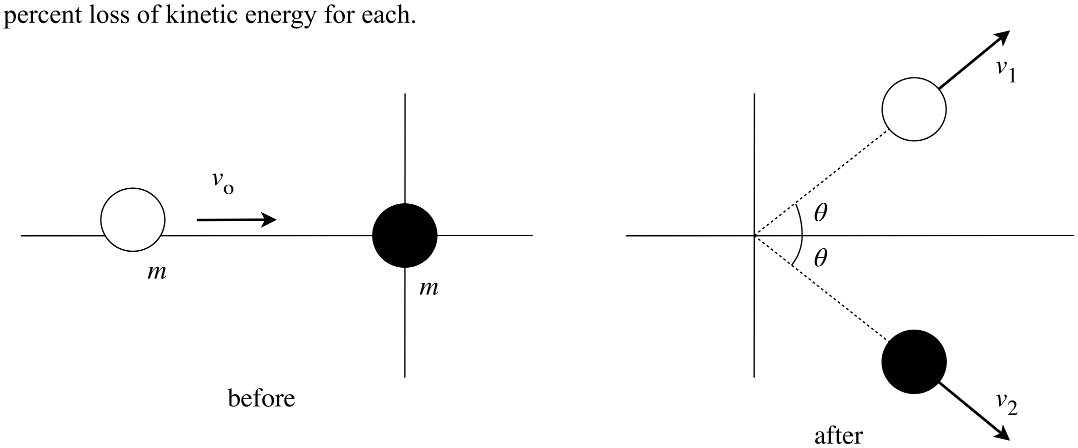
1.A 2.D 3.C 4.A 5.C 6.B 7.D 8.B 9.E 10.D 11.E 12.C 13.B 14.E 15.C

Part II
Show your work
Credit depends on the quality and clarity of your explanations

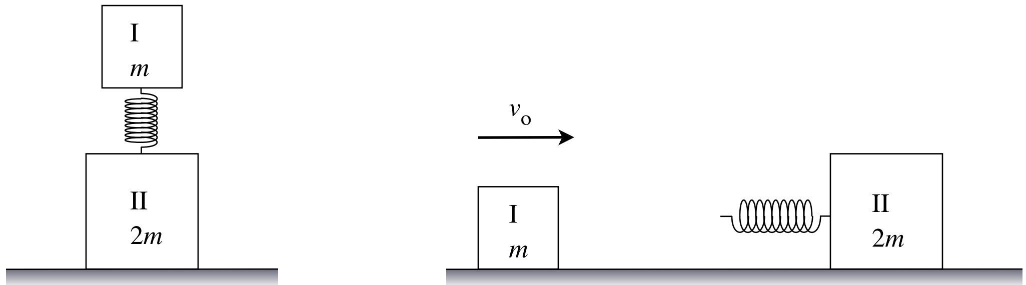
1. On a frictionless ice block a disk of mass m moving with speed v_o collides with a second disk, also of mass m, initially at rest. It is not known whether the collision is elastic. After the collision the disks move as shown below, right; each disk makes an angle θ with the original line of motion. In terms of v_o and θ find:

   A) the speeds v_1 and v_2 of the two disks after the collision
   B) the velocity of the center of mass of the system after the collision
   C) the percent loss of kinetic energy of the system

   D) Using part C) above, determine the smallest and largest values of θ for a physically possible collision, and the percent loss of kinetic energy for each.



2. Two blocks I and II shown below, left have masses m and $2m$, respectively. Block II has an ideal massless spring attached to one side. When block I is placed on the spring as shown, the spring is compressed a distance D at equilibrium. Express your answer to all parts of the question in terms of the given quantities and physical constants.

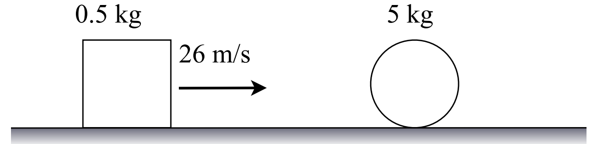


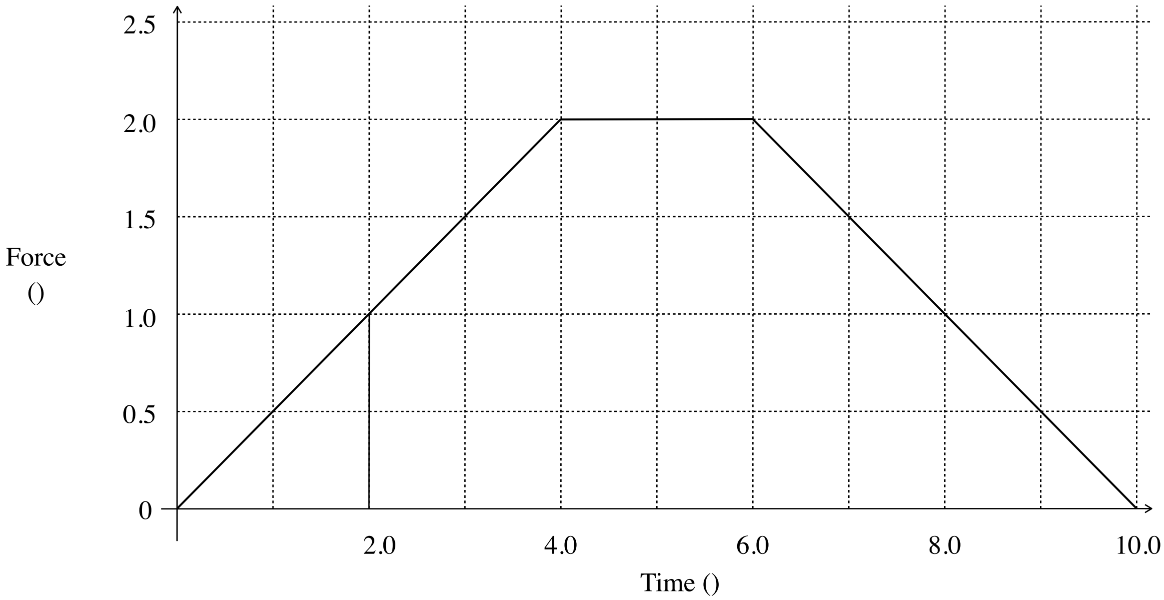
A) Determine the stiffness constant k of the spring

Later the two blocks are on a frictionless horizontal surface. Block II is stationary and block I approaches with speed v_o as shown above, right.

B) Determine the maximum compression of the spring during the collision.
C) Determine the velocities (magnitude and direction) of both blocks when block I has again separated from the spring.

3. A 5 kg ball is initially at rest on a horizontal frictionless surface as shown to the right. A hard plastic cube of mass 0.5 kg slides across the surface at a speed of 26 m/s and strikes the ball. The figure below shows a graph of the force exerted on the ball by the cube as a function of time.





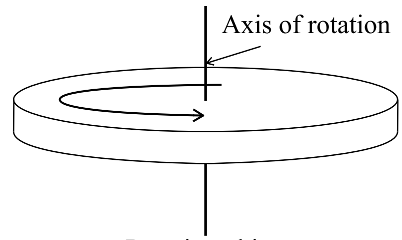
A) Determine the total impulse given to the ball.
B) Determine the horizontal velocity of the ball immediately after the collision.
C) Determine speed and direction of travel (if any) of the cube immediately after the collision.
D) Determine the kinetic energy dissipated in the collision.

AP Physics C
Unit 6 Name______________________________

We now turn our attention to rotational motion, which for simplicity we have ignored so far. Recall that a rigid object in three dimensions has six *degrees of freedom*; three of translation (in the x, y, and z axes), and three of rotation (about each of the axes). We now ignore translation for the time being and concentrate on pure rotation, first about one axis, then the more general case.

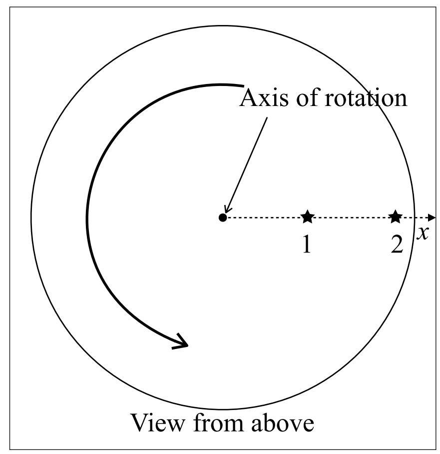
Angular Kinematics

For a particle moving in one dimension (see Unit 1, page 1.2) we defined two fundamental quantities, *position* and *time*, symbolized by the variables x and t. We then used calculus to derive *velocity* and *acceleration*, v and a, and produced the equations of motion for constant and non-constant acceleration. Using this as a model, we can develop a similar description of an object rotating about a fixed axis.



Rotating object

The diagram to the right is a view of the rotating object from above at an arbitrary time $t = 0$, where we have chosen an x-axis that remains stationary as the object rotates. Two paint spots have been made on the object; both are on the x-axis at $t = 0$. Spot 1 is a distance r_1 from the axis of rotation, and spot 2 is a distance $r_2 = 2r_1$ away.



A) On the diagram, mark where these spots would be a short time Δt later.
B) For both spots, indicate the path they traveled during the time Δt. Label the path lengths s_1 and s_2.
C) What is the relationship between the length of s_1 and the length of s_2?
D) What is the relationship between the ratio $\frac{s_1}{r_1}$ and the ratio $\frac{s_2}{r_2}$?

We wish to define a parameter analogous to *position*, that will indicate the *orientation* of the rotating object at any time. Position is defined with respect to an *origin* ($x = 0$), and similarly we define orientation with respect to some reference orientation, such as the orientation at $t = 0$. We define the *angular position* θ as the ratio s/r for any point on the rotating object with respect to the original orientation. What are the SI units of angular position?

Angular position is also measured in more familiar units, such as *degrees* (°) or *revolutions* (rev). The SI units of angular position are given the name *radians* (rad). These units are related by: 1 rev = ______ ° = ______ rad.

E) On the diagram above, draw velocity vectors using a consistent scale for spots 1 and 2 at $t = 0$ and at $t = \Delta t$.

Consider the velocity vectors for each point on the rotating object. Remember that the object may have some thickness, as in the first diagram above.

F) Describe the geometric shape of a set of points for which the velocity vectors have the *same magnitude*.
G) Describe the geometric shape of a set of points for which the velocity vectors have the *same direction*.
H) Describe a set of points for which the velocity vectors have the *same magnitude and direction*.

We wish to define a parameter analogous to *velocity*, that will indicate how fast the object is rotating. Recall that velocity is *rate of change of position*: $\frac{dx}{dt}$. We define *angular velocity* as *rate of change of* ____________________: $\frac{\quad}{dt}$

We use the symbol ω (the Greek letter omega, not w, the Latin letter double-u) for angular velocity. List three different units for angular velocity, including the SI units: A commonly used unit of angular velocity is *rpm*. What does rpm stand for?

We wish to define a parameter analogous to *acceleration*, that will indicate how the object's angular velocity is changing. Recall the definition of acceleration and write a similar definition of *angular acceleration*:

rate of change of ____________________: $\frac{\quad}{dt}$

We use the symbol α (the Greek letter alpha) for angular acceleration. List three different units for angular acceleration, including the SI units:

We can write equations of rotational motion similar to the equations for linear motion. Start by writing the defining equation for angular acceleration as a *differential equation*: The goal is to write equations that tell us the angular position (θ) of the object, and how fast it's rotating (ω) at any time (t), given only the initial conditions (its initial orientation θ_0 and angular velocity ω_0) As usual, we will choose the initial time to be $t = 0$.

Take the definite integral of both sides of the equation of motion, choosing appropriate limits of integration. The lower limits should represent the initial conditions, and the upper limits should represent conditions at time t:

Can either or both of these integrals be evaluated? If so, evaluate; if not, explain why not.

Angular Kinematics with Constant Angular Acceleration

Assuming that acceleration is constant, evaluate the integrals on both sides of the equation above, and solve for ω as a function of time.

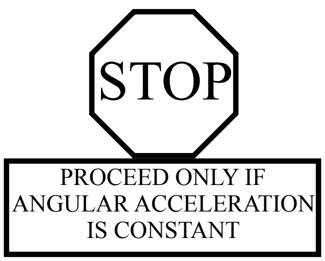
Write the equation you just derived as a differential equation in θ with the variables separated.

Choose appropriate limits of integration and integrate once again to obtain an equation for θ as a function of t.

Combine the equations above for $\omega(t)$ and $\theta(t)$ to eliminate α.

Combine the equations above for $\omega(t)$ and $\theta(t)$ to eliminate t.

Complete the table to the right, which is similar to Table 10-1 on p. 247, by filling in the four kinematic equations you derived above. *Memorize this table!*



Variable left out	Kinematic Equation for Constant Angular Acceleration
θ	
ω	
α	
t	

Remember that these equations apply *only* when angular acceleration is constant. For the following, assume constant angular acceleration. When showing your solution, make a table of variables, write the appropriate equation, solve and plug in.

Example

A CD player spins the CD at different speeds depending on whether the track being played is toward the center of the disc or closer to the edge. We will assume the following approximate figures: 530 rpm for the inner track and 240 rpm for the outer track. In order of normal playback, the tracks go from inner to outer.

A) When you first put a CD into certain players, the disc spins up so that the player can read the track/time information. Assuming that this information is on the outer track, and that it takes a certain player 2 seconds to spin up, find the angular acceleration of the disc in rad/s^2.

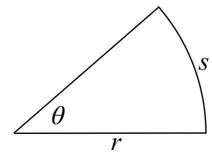
A player is programmed to play the first (inner) track, then the last (outer) track. The time between tracks is 3 seconds.

B) Calculate the angular acceleration of the disc as it switches between tracks, in rad/s^2.

C) Calculate the number of revolutions the disc makes during this time.

The reason why the CD player changes the rotational speed of the CD is so that the laser pickup "sees" the surface moving by at the same linear speed regardless of which track is being read. It's convenient to be able to relate the angular displacement, speed and acceleration to the corresponding linear ones for a rotating object.

Recall (page 6.1) the definition of angular position. We can write this as $s = r\theta$. *Provided we measure angles in radians*, we can calculate the linear displacement of a point on a rotating object if we know its angular displacement and the distance to the axis of rotation.



Take the time derivative of both sides of this equation: Write this in terms of the linear speed v and the angular speed ω: *Provided we measure angles in radians*, we can calculate the linear speed of a point on a rotating object if we know its angular speed and the distance to the axis of rotation.

Take the time derivative of both sides again: Write this in terms of the tangential acceleration a_t and the angular acceleration α: *Provided we measure angles in radians*, we can calculate the tangential acceleration of a point on a rotating object if we know its angular acceleration and the distance to the axis of rotation.

In addition, we can express the centripetal acceleration of a point on a rotating object in terms of the angular velocity and the distance to the axis: $a_c =$

Example

The innermost track of a CD is at a radius of about 2.5 cm, and the outermost is at about 5.5 cm.

A) Find the tangential speed the laser pickup "sees" when reading the inner track.

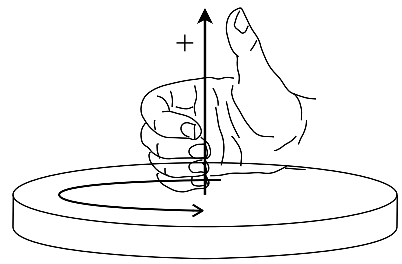
B) Find the tangential speed the laser pickup "sees" when reading the outer track.

C) Find the tangential acceleration of a point on the outer track when the disc first spins up (p. 6.3).

D) Find the centripetal acceleration of a point on the outer track when that track is being read.

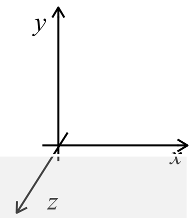
Rotational Kinematics in Vector Form

In linear kinematics, we first discussed the one-dimensional case, then extended the same concepts to 2 and 3 dimensions using vectors. Linear displacement, velocity and acceleration are all vector quantities. If we want to use vectors to describe rotation in more than one dimension (ie., about more than one axis), then we have to be sure that the rotational quantities behave like vectors.



Take angular displacement, for example. To consider it a vector quantity we have to agree on what we mean by *direction*. It's easy to associate a unique straight line with any rotating object, namely the axis of rotation. This provides a simple way to define the direction of the rotational variables; all we have to agree on is which direction should be positive and which should be negative.

By convention, we use the Right-Hand Rule (RHR) to choose the positive direction. Imagine grasping the axis of rotation with the fingers of your right hand curling around in the same way the object rotates. The direction of your thumb is the positive direction.



One of the important properties of vectors is that they obey the commutative law under addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (ie., it doesn't matter in which order you add vectors). Let's choose axes as follows: x-axis points to your right, y-axis points up, and z-axis points toward you. Place your calculator on the desk in front of you face up, so that the keys and screen point in the positive y-direction. Have the person sitting next to you do the same. The calculators are going to undergo two positive angular displacements: 1) 90° about the x-axis and 2) 90° about the y-axis. One of you do 1) then 2), and the other do 2) then 1). After these rotations, what direction do the keys and screen point?

1) then 2): 2) then 1): This shows that angular displacement is *not* a vector quantity. Now try the same exercise, but this time make your rotations about 10° instead of 90° (just estimate). Compare the final positions of the calculators:

The smaller the angular displacements are, the closer they are to obeying the commutative law. Consider a similar example: From a given starting point, imagine walking 1 meter north, turning 90° right, walking another 1 meter, turning 90° right, and walking 1 more meter. Where do you end up with respect to your starting point? Now imagine following the same instructions on a larger scale: Start on the equator of the earth, and instead of walking 1 meter each time, walk 1/4 of the earth's circumference. Where do you end up with respect to your starting point?

When you walk 1 meter, the *angular displacement* with respect to the center of the earth is very small; when you walk 1/4 of the earth's circumference, it's not.

In the limit, differentially small angular displacements $d\theta$ obey the commutative law, and in fact behave like vectors in all other respects as well. If $\overrightarrow{d\theta}$ is a vector, then $\dfrac{\overrightarrow{d\theta}}{dt}$ is as well, since it is a vector divided by a scalar. So angular velocity $\vec{\omega}$ is a vector, and so is its time derivative, angular acceleration $\vec{\alpha}$.

Example

Consider an object rotating so that the angular velocity vector $\vec{\omega}$ points in the positive y-direction, as in the diagram above.

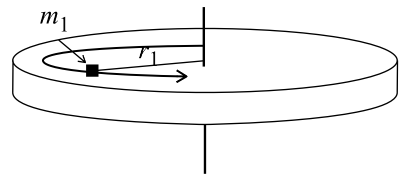
A) If the object is made to spin faster, what is the direction of the angular acceleration vector $\vec{\alpha}$?

B) If the object is slowing down, what is the direction of the angular acceleration vector $\vec{\alpha}$?

Linear dynamics is based on Newton's laws, which are expressed in terms of force, mass and acceleration. Newton's laws for rotational dynamics are expressed in terms of angular acceleration, but we need to find suitable counterparts for force and mass.

Rotational "Mass"

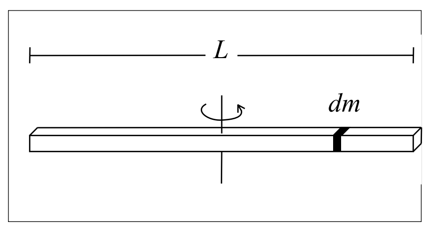
We can think of mass as *translational inertia*; it is a measure of how hard it is to accelerate something translationally. *Rotational inertia* is a measure of how hard it is to give something an *angular* acceleration; ie., to change its rate of rotation. It depends not only on the mass but on the mass *distribution*, or how far from the axis of rotation the mass is. We can make this notion precise by considering the kinetic energy of a rotating body.



Imagine an object spinning about a fixed axis, and consider a small piece of mass m_1 that's a distance r_1 from the axis. Write the kinetic energy of this piece in terms of m_1, r_1, and the angular speed ω:

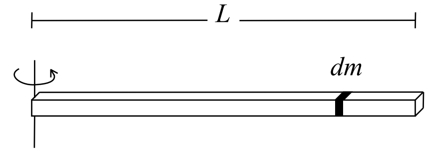
Now consider the entire spinning object. Its kinetic energy will be the sum of all (say N) such pieces. Using summation notation, write the total kinetic energy of the object: Factor out common terms so that your expression has the form: $\frac{1}{2}(?)\omega^2$: This is the expression for the kinetic energy of a rotating object; the expression in parentheses is the rotational version of mass, called the *rotational inertia*, symbolized by I. So for a finite collection of N objects, write the expression for the rotational intertia: $I =$ We can consider any rotating object to be divided into an infinite number of pieces of mass dm, each a certain distance r from the axis of rotation. Use integral notation to write the expression for the rotational inertia of an object about a fixed axis: $I =$ Note that the same object can have different values for I depending on the chosen axis of rotation.

Examples



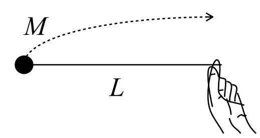
1. Consider a thin stick of length L and mass M, rotating about an axis through its center as shown to the right. A differentially small piece of the stick of mass dm is shown.
   A) Choose parameters that will identify the position and size of dm and label them in the diagram.
   B) In terms of your parameters, find the ratio $\frac{dm}{M}$:
   C) Substitute into the integral expression for I, choose appropriate limits and integrate to find I in terms of M and L:

D) Use a similar method to calculate the rotational inertia about one end:

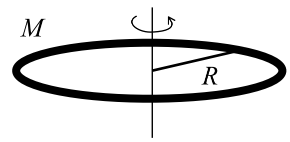


E) A uniform door of mass M has a height H and width W. It has hinges on one side. What is the rotational inertia of the door? Explain your reasoning rather than performing an integration.

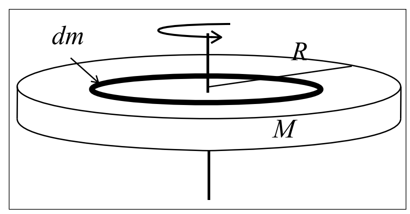
2. A small ball of mass M is tied to a light string of length L and whirled in a horizontal circle as shown to the right. State the rotational inertia of the ball and explain how you can determine it from the definition without performing the integration.



3. A hoop of mass M and radius R is rotated about its axis as shown to the right. State the rotational inertia of the hoop and explain how you can determine it from the definition without performing the integration.



4. A thin disk of mass M, and radius R is rotated about its axis as shown to the right. A ring of mass dm is shown.



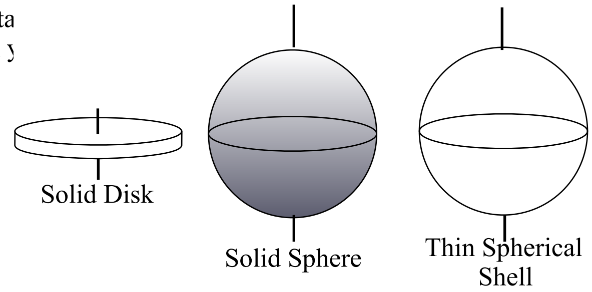
A) Choose parameters that will identify the position and size of dm and label them in the diagram.

B) In terms of your parameters, find the ratio $\dfrac{dm}{M}$:

C) Substitute into the integral expression for I, choose appropriate limits and integrate to find I:

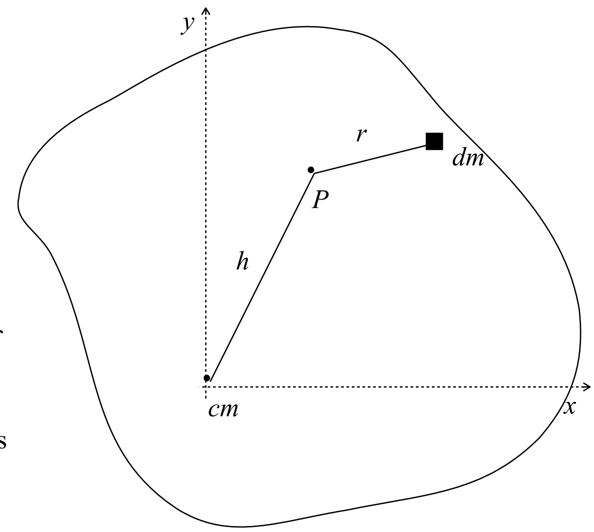
D) A solid cylinder of mass M, radius R and height H is rotated about its axis. What is its rotational inertia? Explain without performing an integration.

5. The shapes to the right have the same mass and radius, and are rota their rotational inertias in order from least to greatest, and explain y



Parallel Axis Theorem

Consider an arbitrarily-shaped object of mass M with a center of mass at *cm*. Imagine that we are given the rotational inertia about an axis through the center of mass perpendicular to the page, I_{cm}. Now consider another point on the object, a distance h from the *cm*. Call this point P. We can express the rotational inertia about a parallel axis through P (I_P) in terms of I_{cm}, M and h.



To prove this theorem, consider a small mass dm, a distance r from point P. Express I_P as an integral over all such dm's:

$I_P =$

On the axes whose origin is at the center of mass, let the coordinates of dm be (x, y), and the coordinates of P be (a, b). Express the integrand from the above expression for I_P in terms of x, y, a, and b instead of r.

$I_P =$

Expand this expression and collect them into three integrals; one that contains just x's and y's, one that contains just a's and b's, and one that contains a mixture of a's, b's, x's and y's:

$I_P =$

Consider the first group above. Re-express this group in terms of I_{cm} and explain your reasoning.

Consider the second group above. Re-express this group in terms of M and h and explain your reasoning.

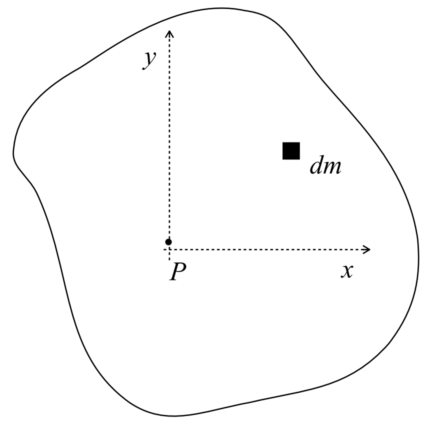
Now consider the remaining terms. Use the definition of center of mass to show that these terms are equal to zero. Explain.

Finally, state the parallel axis theorem, which gives I_P in terms of I_{cm}, M and h: $I_P =$

Example

Use the parallel axis theorem to determine the rotational inertia of a stick about one end, given the rotational inertia about its center (page 6.6). Show that your results agree with the calculated value on page 6.7.

Perpendicular Axis Theorem



Consider an arbitrarily-shaped planar object of mass M. We have chosen an arbitrary point P and drawn x and y axes with P at the origin. A small dm is shown with coordinates (x, y). Write the indefinite integral that expresses the rotational inertia of the object about an axis through P, parallel to the y axis: $I_y =$

Write the indefinite integral that expresses the rotational inertia of the object about an axis through P, parallel to the x axis: $I_x =$ Write the indefinite integral that expresses the rotational inertia of the object about an axis through P, parallel to the z axis: $I_z =$

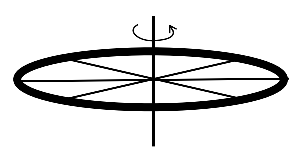
The *Perpendicular Axis Theorem* relates I_z to I_x and I_y: $I_z =$

Examples

1. A coin (a thin disk) of mass M and radius R is spun on a table top about a diameter. Use the Perpendicular Axis Theorem and symmetry arguments to find the rotational inertia about this axis.

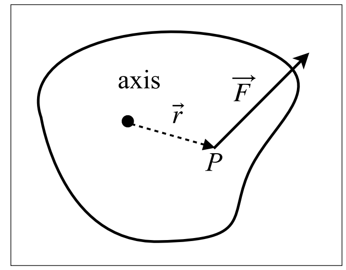
2. A thin rectangular sheet of mass M, length L and width W is rotated about an axis through its center, perpendicular to the sheet. Use the Perpendicular Axis Theorem and symmetry arguments to find the rotational inertia about this axis.

Rotational "Force"



To get something with rotational inertia to have angular acceleration, we need the rotational version of *force*. Imagine spinning a bicycle wheel. You grab the edge and exert a force *tangent* to the edge to get it to spin. If you exert the same force *perpendicular* to the edge (toward or away from the axle) it would have no effect.

The rotating effect of a force depends on the magnitude of the force (greater force will cause greater angular acceleration), but it also depends on the direction of the force relative to the axis of rotation.



The diagram to the right represents an arbitrarily shaped object pivoted on an axis perpendicular to the page. A force $\vec{F}$ is applied at point P, and the position vector of point P with respect to the axis is $\vec{r}$.

A) With a ruler or straightedge, extend the line of the force to the lower left, and draw the perpendicular distance from this line to the axis.

This distance is called the *lever arm* of the force with respect to this axis, symbolized by ℓ. The rotational effect of the force is proportional to the lever arm as well as the magnitude of the force. The rotational effect of a force is called *torque*, symbolized by τ (the Greek letter tau). It is the product of the magnitude of the force and the length of the lever arm:

$$\tau = F\,\ell$$

B) On the diagram, there is a right triangle with ℓ as one side. Label the angle opposite this side θ.

C) Express the torque in terms of θ and the magnitudes of $\vec{F}$ and $\vec{r}$: $\tau =$

D) $\vec{F}$ and $\vec{r}$ are vectors, and so is torque. The direction of the torque is the same as the direction of the angular acceleration it causes. For the object in the diagram above, what is the direction of the torque, in terms of the standard axes (page 6.5)?

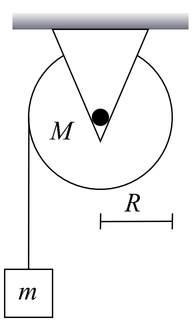
E) Use vector notation to express the torque vector in terms of $\vec{F}$ and $\vec{r}$: $\vec{\tau} =$

If there are multiple forces on an object, each may cause a torque about a given axis, and the net torque is the vector sum of these torques. When we draw free-body diagrams involving torque, we draw a complete representation of the object, and draw the forces where they actually apply, so that we can determine their lever arms. These are called *extended* free-body diagrams. By analogy with Newton's Second Law for translation, we can now write Newton's Second Law for rotation:

$$\Sigma\vec{\tau} =$$

A given force can therefore play a role in both translation and rotation. In systems involving both, we use both versions of Newton's Second Law.

Examples

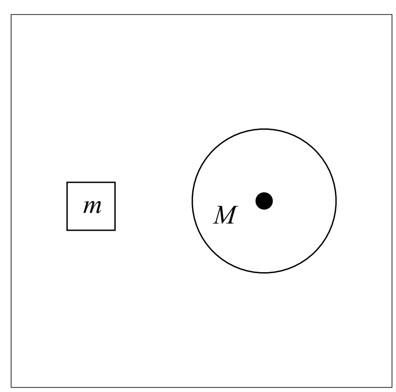


1. A uniform disk of mass M and radius R is mounted on frictionless bearings and held fixed. A light string is wrapped around the disk and tied to a mass m that is released from rest as shown to the right.

   A) In the space below, right, draw a free-body diagram showing the forces acting on the mass and an extended free-body diagram of the disk once the mass is released.

   B) Write the equations that result from applying Newton's Second Law for translation to each of the two objects:

   C) Write the equation that results from applying Newton's Second Law for rotation to the disk:



   D) A student makes the following statement: "The torque on the disk is equal to the weight of the mass, mg, times the lever arm, which is R." What is wrong with this statement?

E) How is the angular acceleration of the disk related to the tangential acceleration of a point on the rim of the disk (see page 6.4)?

F) How is the tangential acceleration of a point on the rim of the disk related to the linear acceleration of the hanging mass? Explain.

G) Your equations on the previous page involve a, the linear acceleration of the hanging mass, and α, the angular acceleration of the disk. Explain how these variables are related *in this case*:

H) You calculated the rotational inertia of a disk on page 6.7. Combine this and the equations from parts B), C), and G) to find the linear acceleration of the hanging mass.

I) Find the tension in the string while the mass is falling.

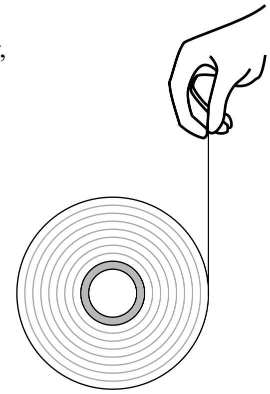
After the hanging mass has fallen a distance h, it is moving with a speed v, and the disk is rotating with angular speed ω.

J) How are v and ω related? Explain.

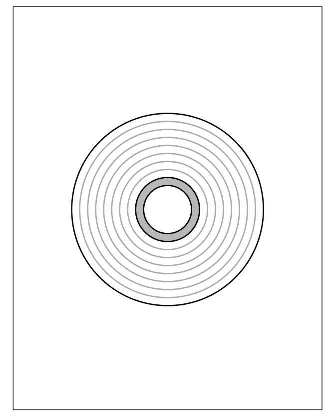
K) Use conservation of energy to find the speed v of the hanging mass.

L) Show by using kinematics that an object accelerating at the rate found in part H) will be going at the speed calculated above after moving a distance h.

2. A roll of paper towels with mass M and radius R is dropped from rest while the end of the roll is held fixed above the edge of the roll. Ignore the inner radius and treat the roll as a solid cylinder, and assume that the paper is very thin compared to R.



A) In the space below, draw an extended free-body diagram of the roll as it is dropping.



B) As the roll drops, will its center of mass *curve left*, *curve right*, or *drop straight down*? Explain your reasoning.

C) As the roll falls, how is its linear acceleration a related to its angular acceleration α? Explain *using the center of mass frame of reference*.

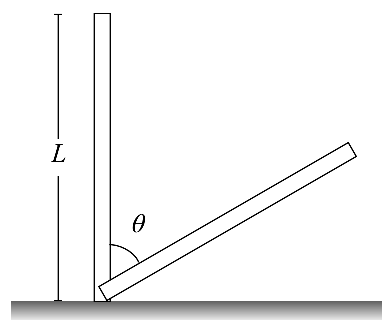
D) Find the linear acceleration of the roll as it falls.

Rotational "Work"

A force does work given by $W = \int \vec{F} \cdot \vec{dr}$. Write the rotational version of this statement.

Example

A stick of mass M and length L stands vertically on a table. It is given a slight push and falls to the table. The diagram to the right shows the stick after it has fallen through an angle θ.



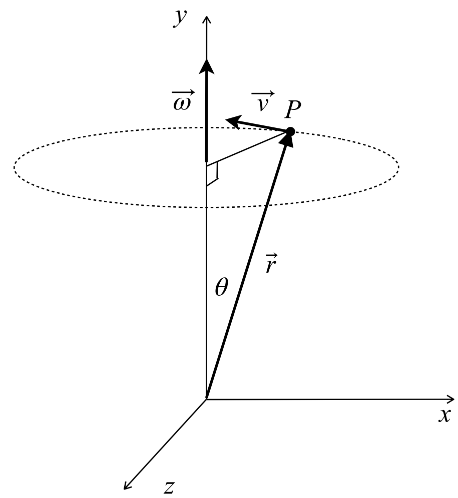
A) Determine the torque about the point of contact with the table at the instant shown.

B) By integrating with appropriate limits, find the work done by the torque by the time the stick hits the table, and compare this to the loss of gravitational potential energy of the stick.

C) Find the angular speed of the stick as it hits the table.

Rotation in General Vector Form

The particle at point P in the diagram to the right is part of a rigid object rotating about a fixed axis in the y-direction. The angular velocity of the object is represented by the vector $\vec{\omega}$, in the positive y-direction. The position of P relative to the origin is represented by the vector $\vec{r}$, which makes an angle θ with the y-axis. The tangential velocity of the particle is represented by the vector $\vec{v}$. In the general case, none of these vectors is necessarily constant.



A) What is the radius of the particle's circular path?

B) How are the magnitudes ω and v related (see page 6.4)?

C) Since $\vec{r}$, $\vec{v}$, and $\vec{\omega}$ are vectors, the above relationship can be written as a vector cross product:

D) Take the time derivative of both sides of the above equation, using the product rule and remembering that vector cross products are *not* commutative:

E) Write this equation using only the vectors $\vec{a}$, $\vec{\alpha}$, $\vec{\omega}$, $\vec{v}$, and $\vec{r}$:

F) Show that the equation in part E) is equivalent to: , where $\vec{a}_t$ is the tangential acceleration vector and $\vec{a}_c$ is the centripetal acceleration vector.

AP Physics C
Unit 6 Practice Quiz

Name ____________________

Part I
Multiple Choice

1. One rpm is about

   A) 0.0524 rad/s
   B) 0.105 rad/s
   C) 0.95 rad/s
   D) 1.57 rad/s
   E) 6.28 rad/s

2. Of the following, the equation that is valid only when the angular measure is expressed in radians is

   A) $\alpha = \frac{d\omega}{dt}$

   B) $\omega = \frac{d\theta}{dt}$

   C) $\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$

   D) $\omega = \frac{v}{r}$

   E) $\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

3. The angular speed of the second hand of the classroom clock is

   A) $\pi/1800$ rad/s
   B) $\pi/60$ rad/s
   C) $\pi/30$ rad/s
   D) 2π rad/s
   E) 60 rad/s

4. Ten seconds after the switch of an electric fan is turned on, the fan rotates at 300 rpm. Its average angular acceleration is

   A) 3.14 rad/s^2
   B) 30 rad/s^2
   C) 30 rev/s^2
   D) 50 rev/min^2
   E) 1800 rev/s^2

5. If the angular velocity vector of a spinning body points in a direction opposite to that of the angular acceleration vector, then

   A) the body is spinning faster
   B) the body is starting to spin in the opposite direction
   C) the body is spinning slower
   D) the axis of rotation is changing orientation
   E) none of the above

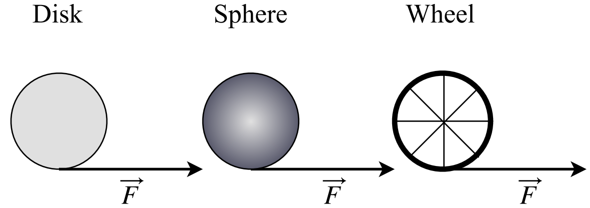
6. A wheel 4 m in diameter rotates with constant angular acceleration of 4 rad/s^2. The wheel starts from rest at $t = 0$ where the position vector to point P makes a 45° angle with the x axis. The expression for the angular position of point P at an arbitrary time t is

   A) 45°
   B) 45° + (2°/s^2)t^2
   C) 45° + (114.6°/s^2)t^2
   D) (4°/s^2)t^2
   E) (229.2°/s^2)t^2

7. String is wrapped around a cylinder of 5.0 cm radius which is free to rotate on its axis. The string is pulled out at a constant rate of 10 cm/s and does not slip on the cylinder. The angular velocity of the cylinder is

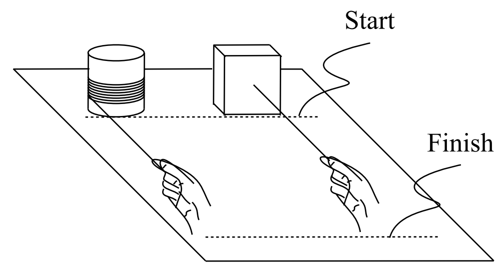
A) 2 rad/s
B) 5 rad/s
C) 10 rad/s
D) 25 rad/s
E) 50 rad/s

8. A uniform disk, a uniform sphere, and a bicycle wheel, all with the same mass and same outer radius, are each free to rotate about a fixed axis through its center. With all objects starting from rest, identical forces are simultaneously applied to the rims as shown. Rank the objects according to their rotational kinetic energies after a given time *t*, from least to greatest.



A) disk, wheel, sphere
B) sphere, disk, wheel
C) wheel, sphere, disk
D) disk, sphere, wheel
E) wheel, disk, sphere

For questions 9 - 10: A spool and a block of equal mass are pulled across a frictionless surface as shown to the right. One string is wrapped many times around the spool so that it can unwind, and the other is fastened to the middle of the block. The strings are pulled with equal force, starting at the same time, so that equal tensions are maintained in the strings.



9. Consider the following statements about which object will cross the finish line first:

I The block will cross first because all of the tension force goes into translational acceleration, whereas for the spool some of the force goes into rotational acceleration.
II The block will cross first because all if its energy goes into translational kinetic energy, whereas for the spool some of the energy is rotational.
III The block and spool will cross at the same time because of Newton's Second Law for translation.
IV The block will cross first because the spool will not translate on a frictionless surface, it will only rotate.

The correct statement(s) is (are):

A) I only B) II only C) I and II only D) III only E) IV only

10. Consider the following statements about the work done by the two hands in moving the objects for a given time *t*:

I The work done by the hand pulling the block is greater because the block moves faster than the spool, so it covers more distance..
II The work done by the hand pulling the spool is greater because the hand pulling the spool moves a greater distance.
III The work done by the two hands is equal because the forces are equal and the displacements are equal.
IV The work done by the hand pulling the spool is greater because the spool has only rotational kinetic energy

The correct statement(s) is (are):

A) I only B) II only C) I and II only D) III only E) IV only

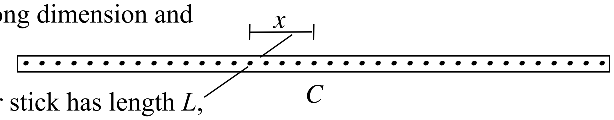
1. B 2. D 3. C 4. A 5. C 6. C 7. A 8. E 9. D 10. B

Part II

Show your work

Credit depends on the quality and clarity of your explanations

1. A meter stick contains many small holes perpendicular to its long dimension and spaced along its length. A student studies rotation of the stick about a stiff wire which can be placed in any one of the holes. The stick can rotate about the wire without friction. The meter stick has length L, mass M, and is released from a horizontal position as shown. The wire is at a variable distance x from C, the center of mass of the meter stick.



A) What is the initial value of the torque τ due to the gravitational force acting on the stick about the axis defined by the wire?

B) The rotational inertia of the stick about C is $ML^2/12$. What is the rotational inertia of the stick about x?

C) What is the instantaneous value of the angular acceleration α_0 of the stick just after it is released?

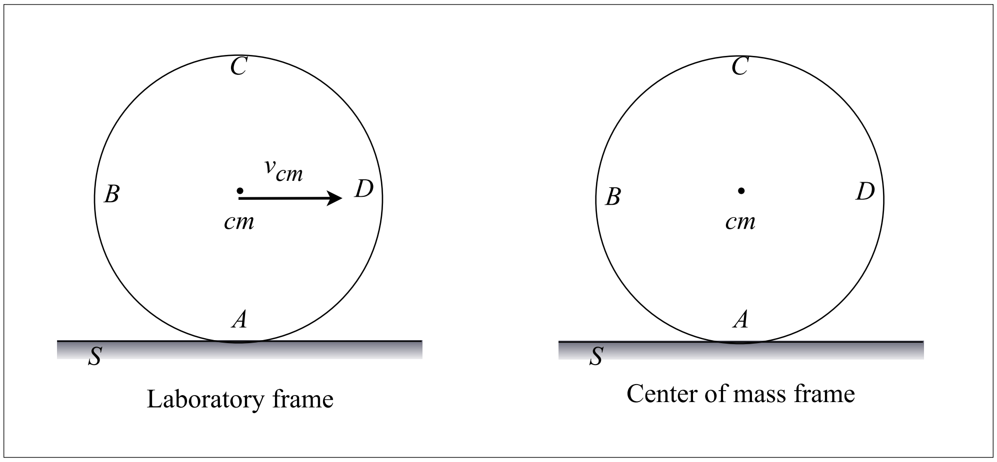
D) Determine the wire location (value of x) for which α_0 (as defined in part C)) is a maximum.

E) For a given x, determine how the value of the instantaneous angular acceleration α depends on the angular displacement θ from its original horizontal position.

F) For the value of x found in part D), find the angular velocity ω of the meter stick after it has rotated to a vertical position.

Combined Rotation and Translation

Consider a wheel of mass M and radius R rolling *without slipping* along a horizontal surface such that the speed of the axle (the center of mass) is v_{cm} in the laboratory frame of reference, as shown below.



A) In each frame, use the same scale to draw vectors representing the velocities of each of the four lettered points on the rim of the wheel, as well as the center of mass and the surface S. Indicate the magnitude of each vector relative to v_{cm}; if a vector at any point has zero magnitude, state that explicitly.

B) Explain how the assumption that the wheel is *not slipping* is reflected in both frames in terms of the vectors you have drawn above.

C) Use the diagram above to explain the relationship between the angular speed ω of the wheel, the linear speed v_{cm} of the center of mass, and the radius R of the wheel *when it rolls without slipping*.

D) Assume the wheel accelerates, still without slipping. Derive an expression for the linear acceleration a_{cm} of the center of mass in terms of the angular acceleration α of the wheel and the radius R.

Assume that the wheel has a rotational inertia I_{cm} about the center of mass. In the laboratory frame, we can think of the wheel as instantaneously being in pure rotation about the point of contact with the surface (point A).

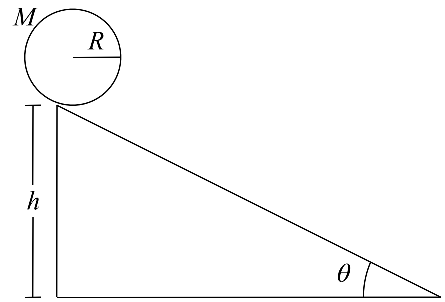
E) What is the angular speed ω of the wheel about the point of contact with the surface in terms of v_{cm} and R?

F) Use the parallel axis theorem to find the rotational inertia of the wheel about the contact point (I_A).

G) Express the rotational kinetic energy of the wheel about point A in terms of I_{cm}, M, R and ω.

H) Use the expression in part G) to show that for any object that is rolling without slipping, the total kinetic energy is the sum of the rotational kinetic energy about the center of mass (pure rotational) and the translational kinetic energy of the center of mass (pure translational).

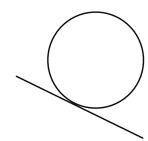
Examples



1. A solid cylinder of mass M and radius R rolls down an incline of height h. The coefficient of static friction is sufficient to prevent the cylinder from slipping.

A) Explain why a frictional force between the ramp and the cylinder is necessary if it is to roll without slipping.

B) In the space to the right, draw an extended free-body diagram of the cylinder as it rolls down the incline. Take care to accurately represent the point of application of each force.



C) Choose appropriate axes and resolve all forces along those axes.

D) Write the equations that result from applying Newton's Second Law for translation.

E) Write the equation that results from applying Newton's Second Law for rotation.

F) Solve the equations above to find the acceleration of the cylinder down the incline.

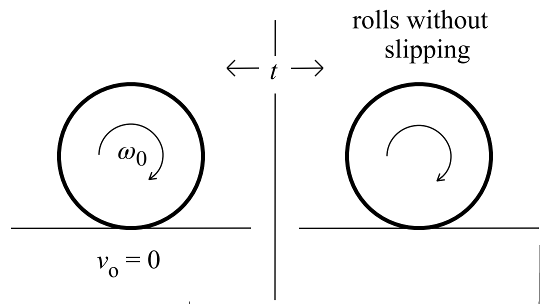
G) Determine the minimum coefficient of friction such that the cylinder will roll without slipping.

H) Use kinematics to find the speed of the cylinder when it gets to the bottom of the incline.

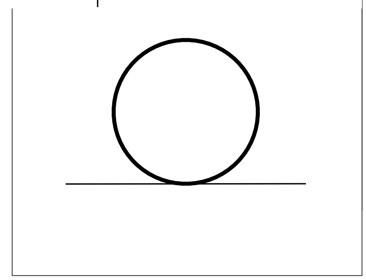
I) Find the total kinetic energy of the cylinder at the bottom of the incline.

J) Find the work done by the frictional force as the cylinder rolled to the bottom of the incline. Explain why your answer makes sense.

2. A hula-hoop of radius r is spun with angular velocity ω_0 in such a way that it hovers briefly above the ground. As it settles to the ground it skids, picking up translational speed as it loses rotational speed. After a time t, it begins rolling without slipping.



A) On the diagram below, right, draw vectors representing the forces acting on the hula-hoop while it is slipping on the ground.



B) Assuming a coefficient of kinetic friction μ_k, find the translational acceleration of the center of mass of the hoop.

C) Use linear kinematics to find the speed v at which it rolls without slipping.

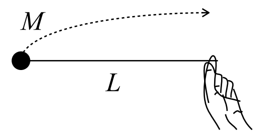
D) Find the angular acceleration of the hoop while it is slipping.

E) Use angular kinematics to find the angular velocity once it rolls without slipping.

F) Find the linear and angular speeds at which it rolls without slipping, in terms of r and ω_0.

G) Find the linear and angular displacements during the time t while it slips, in terms of μ_k, r and ω_0.

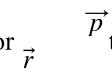
Rotational "Momentum"



Consider again a small ball moving in a circle of radius L at a speed v. Write the expression of the linear momentum $\vec{p}$ of the ball: $\vec{p} =$ The rotational version of momentum is called *angular momentum*, symbolized by $\vec{\ell}$. Write the expression for the angular momentum of the ball: $\vec{\ell} =$

How is the magnitude of the angular momentum ℓ related to the magnitude of the linear momentum p? (Recall the rotational inertia of the ball, page 6.7.)

The linear momentum of the ball above is always perpendicular to the string. In general, the linear momentum can point in any direction with respect to the pivot, as shown to the right. The vector $\vec{r}$ is the position vector of the object with respect to the pivot.



A) On the diagram, draw the *lever arm* of the momentum vector with respect to the pivot, as you did for torque (page 6.10).

In general the angular momentum is the product of the linear momentum and the lever arm. The direction of the angular momentum is determined by the Right-Hand Rule.

B) What is the direction of the angular momentum of the object in the drawing, with respect to the standard axes (page 6.5)?

C) As you did with force and torque, write the angular momentum vector using cross product notation:

$$\vec{\ell} =$$

D) Take the time derivative of both sides of the above equation, using the product rule and remembering that vector cross products are *not* commutative:

$$\frac{d\vec{\ell}}{dt} =$$

E) Write the right side of this equation using only the vectors $\vec{v}$, $\vec{p}$, $\vec{r}$, and $\vec{F}$:

$$\frac{d\vec{\ell}}{dt} =$$

F) One of the two terms in your expression above is zero. Which one is zero, and why?

G) Simplify your expression to show the relationship between angular momentum and torque: $\frac{d\vec{\ell}}{dt} =$

For a system of particles, the total angular momentum $\vec{L}$ is the sum of the individual angular momentum vectors:

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \ldots$$

Take the derivative of both side of this equation: $\frac{d\vec{L}}{dt} =$

Torques due to forces between particles in the system add up to zero. Why?

We can therefore rewrite the equation on the last page as: $\frac{d\vec{L}}{dt} = \sum \vec{\tau}_{\text{ext}}$

So in the absence of external *torques*, *angular momentum* is conserved, just as in the absence of external *forces*, *(linear) momentum* is conserved. We will see situations where angular momentum is conserved but linear momentum is not, and vice versa. According to Emmy Noether's theorem (Unit 4, page 4.5), conservation of angular momentum results from the symmetry that the laws of physics are the same for a system regardless of how it is oriented in space.

Note:

Most rotational quantities are different from their linear counterparts. Force is *not* torque; Angular velocity is *not* linear velocity. However, two quantities are the same in rotation as they are in translation: *Time* and *energy.*

Examples

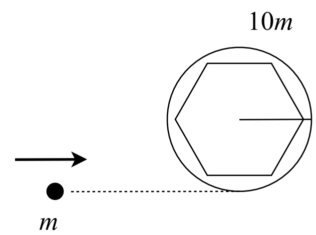
1. Place a meter stick on your desk with the metric side facing up. Tap it with a pen or pencil with a light stroke perpendicular to the stick and parallel to the table top. Pay attention to what the 0 cm side of the stick does when you tap it at various points.
   A) When you tap the stick at the 50 cm mark, does the 0 cm end move in the *same* direction as your tap or the *opposite* direction?
   B) When you tap the stick at the 100 cm mark, does the 0 cm end move in the *same* direction as your tap or the *opposite* direction?
   C) Find by trial and error where to tap the stick so that the 0 cm end doesn't have any instantaneous motion. We'll call this point the "sweet spot." Record your estimate:

For the following, assume that the surface is frictionless (in fact, the interaction is so short that friction doesn't get a chance to do much until after the impulse). Let the duration of the tap be Δt, and the average force during the tap be F. After the stick is tapped, its center of mass moves with speed v, and it rotates with angular speed ω about its center of mass. Call the mass m.

   D) Write the expression that relates the linear impulse to the stick's change in linear momentum.
   E) Let x represent the distance from the end of the stick to where it is tapped. Write the expression that relates the angular impulse to the stick's change in angular momentum.
   F) Since the stick is instantaneously rotating about one end, what is the rotational inertia (see page 6.7)?
   G) Since the stick is instantaneously rotating about one end, what is the relationship between v and ω?
   H) Find an expression for x in centimeters, and compare with your trial-and-error result.

6.24

2. A child of mass m on a playground runs with speed v on a line tangent to the edge of a merry-go-round of radius R. The merry-go-round has a mass 10 m, and is essentially a disk with frictionless bearings initially at rest. The child jumps onto the merry-go-round and holds on.



A) Is linear momentum conserved in this collision? Why or why not?

B) Is angular momentum conserved in this collision? Why or why not?

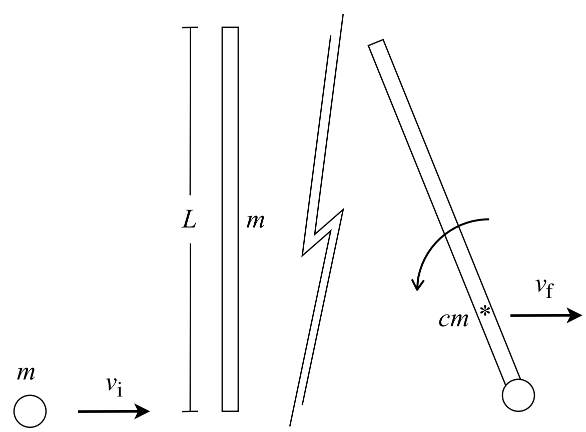
C) Find the final angular speed of the child and the merry-go-round in terms of v and R.

D) Find the energy lost in the process in terms of m and v.

3. A ball of mass m, initially moving with a velocity v_i strikes a bar of length L, also of mass m, at the end of the bar, perpendicular to its length. The surface on which they move is frictionless. The ball sticks to the end of the bar, and the system both rotates and translates as a result.

A) Is linear momentum conserved in this collision? Why or why not?

B) Is angular momentum conserved in this collision? Why or why not?



C) Determine the velocity v_f of the system after the collision.

D) Determine the location of the center of mass of the system with respect to the end of the bar to which the ball is stuck.

E) What is the magnitude and direction of the angular momentum of the ball about this center of mass point, just before the collision?

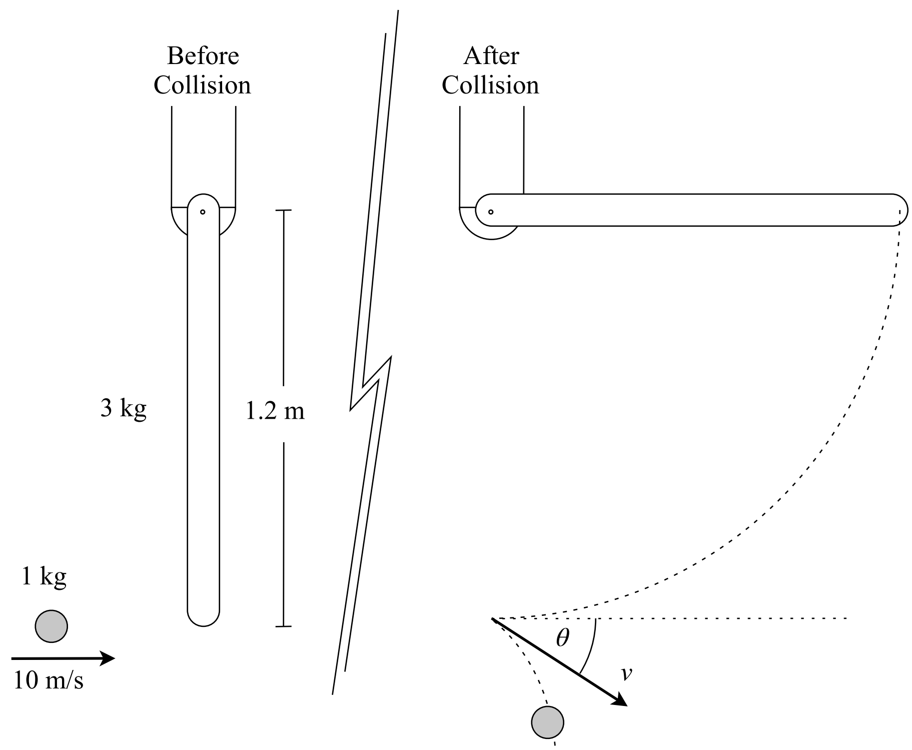
F) The rotational inertia of the bar about its center is $I = \dfrac{mL^2}{12}$. What is the rotational inertia of the bar about the center of mass of the system?

G) Find the rotational inertia of the system about the center of mass, after the collision.

H) Determine the angular velocity ω of the system after the collision in terms of v_i and L.

4. A 1 kg object is moving horizontally with a velocity of 10 m/sec, as shown below, when it makes a glancing collision with the lower end of a bar that was hanging vertically at rest before the collision. The collision is elastic. The bar, which has a length L of 1.2 meters and a mass M of 3 kg, is pivoted about the upper end. Immediately after the collision the object moves with a speed v at an angle θ relative to its original direction as shown. The bar swings freely, and after the collision reaches a horizontal position before swinging back down. The rotational inertia of the bar about the pivot is $ML^2/3$. Ignore all friction, and use g = 10 m/sec^2.

A) Is linear momentum conserved in this collision? Why or why not?



B) Is angular momentum conserved in this collision? Why or why not?

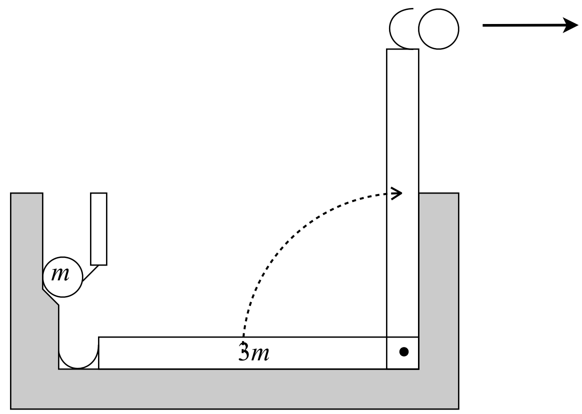
C) Determine the magnitude of the angular momentum of the object about the pivot just before the collision.

D) Determine the angular velocity of the bar immediately after the collision.

E) Determine the speed v of the 1 kg object immediately after the collision.

F) Determine the angle θ.

5. A pitching machine throws balls for batting practice. It has a uniform arm of length L and mass $3m$ that starts at rest horizontally, receiving a ball of mass m in a cup at its end. A spring provides a constant torque τ, which rotates the arm 90°, where it strikes a stop so that the ball goes forward horizontally. The size of the ball is small compared to L.



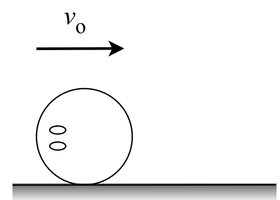
A) Find the rotational inertia of the arm and ball when the ball is in the cup.

B) Find the angular velocity of the arm when it strikes the stop.

C) Find the linear velocity of the ball as it leaves the arm.

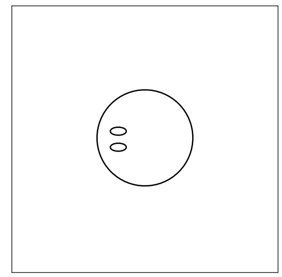
D) Find the ratio of the final kinetic energy of the ball to the torque τ.

6. A bowling ball of mass M, radius R and rotational inertia $\frac{2}{5}MR^2$ is propelled toward the pins on a bowling alley with a coefficient of kinetic friction μ_k. Initially the ball is sliding at a speed v_o and not rotating. Gradually its rotational speed increases until it rolls without slipping.



A) In the space below, draw a free-body diagram of the bowling ball while it is sliding.

B) What is the linear acceleration of the ball while it is sliding?



C) What is the angular acceleration of the ball while it is sliding?

D) At what time t does the ball begin to roll without slipping?

E) For what distance x does the ball slide before it rolls without slipping?

F) What is the speed of the ball once it rolls without slipping?

7. A thin uniform rod of mass M and length L has a rotational inertia about its center of $\frac{1}{12}ML^2$. The rod is glued to a thin hoop, also of mass M, and radius $L/2$ to form a rigid assembly, as shown to the right in Figure I. The centers of the rod and hoop coincide at point P.

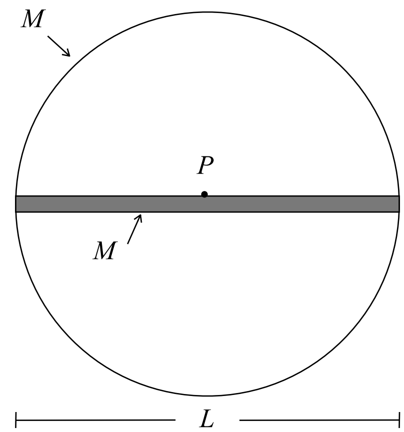


Figure I

A) Determine the rotational inertia of the assembly.

Several turns of string are wrapped tightly around the circumference of the hoop, and the free end is tied to a point far above the floor. The system is released from rest when the string is perfectly vertical, as in the figure below, to the right.

B) Find the initial linear acceleration of the system in terms of g.

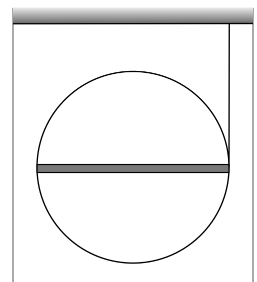


Figure II

C) On the diagram in Figure II, sketch the path of point P after the assembly is released.

Now the system is mounted on a horizontal frictionless axle through point P, perpendicular to the page. The rod is in a horizontal position when a small ball of putty, also of mass M, drops from a height $L/2$ above the end of the rod and sticks to the assembly where the rod and hoop are glued.

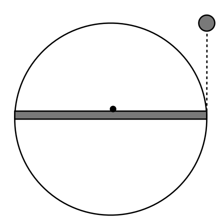


Figure III

D) Find the initial angular velocity of the assembly.

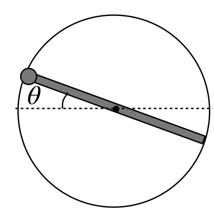
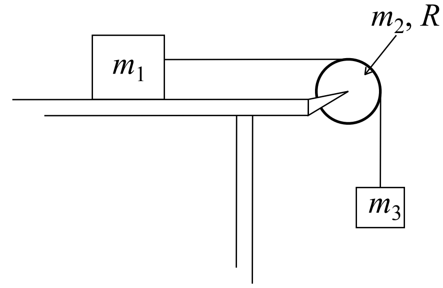


Figure IV

E) Find the angle between the rod and the horizontal when the assembly momentarily stops.

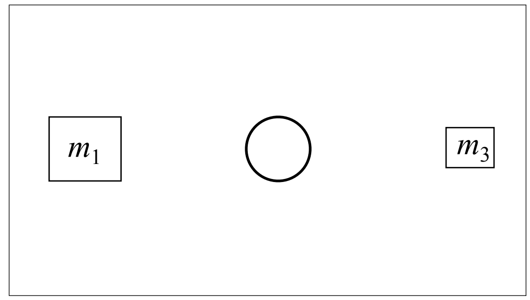
6.29

8. A block of mass m_1 is on a frictionless table, connected by a light thread over a pulley of mass m_2 and radius R to a block of mass m_3 as shown to the right. The pulley is a solid disk, whose axle is held fixed by a support attached to the table. The system is released from rest and the thread does not slip on the pulley.



A) In the spaces provided, draw free-body diagrams of the blocks and the pulley.

B) Is the tension in the horizontal part of the string *greater than*, *less than*, or *equal to* the tension in the vertical part of the string? Explain.



C) Write the equations that result from applying Newton's Second Law for translation to the two blocks.

D) Write the equation that results from applying Newton's Second Law for rotation to the pulley.

E) Determine the acceleration of the system.

Angular Kinematics

Variable left out	Kinematic Equation for Constant Angular Acceleration
θ	$\omega = \omega_0 + \alpha t$
ω	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
α	$\theta = \frac{1}{2}\left(\omega + \omega_0\right) t$
t	$\omega^2 = \omega_0^2 + 2\alpha\left(\theta - \theta_0\right)$

- Angular kinematics is similar to linear kinematics, except that angular displacement (θ) plays the role of linear displacement (x or $\vec{r}$).
- Angular velocity: $\omega = \dfrac{d\theta}{dt}$ Angular acceleration: $\alpha = \dfrac{d\omega}{dt}$
- If angular acceleration is constant, the equations for angular kinematics, similar to those for linear kinematics, can be derived by integrating the differential equation $d\omega = \alpha\, dt$ twice.
- From the definition of radian measure of an angle, we can derive expressions relating linear and angular quantities provided we use radians:

$$s = r\theta, \text{ so } \frac{ds}{dt} = r\frac{d\theta}{dt}, \text{ or } v = r\omega.$$

$$v = r\omega, \text{ so } \frac{dv}{dt} = r\frac{d\omega}{dt}, \text{ or } a = r\alpha. \text{ (tangential acceleration)}$$

Angular Dynamics

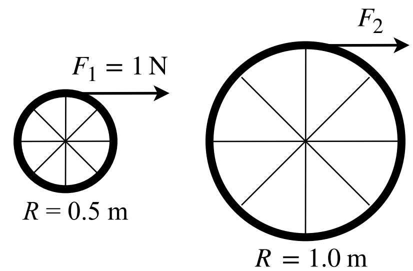
- By analogy with the linear variables Force (F) and Mass (m), angular dynamics introduces the variables Torque (τ) and Rotational Inertia (I).
- Rotational inertia of a point mass m at a distance r from the axis of rotation is $I = mr^2$. The rotational inertia of a hoop of radius r and mass m is also $I = mr^2$.
- In general, the rotational inertia of an extended object is obtained by integrating $I = \int r^2 dm$ over the object.
- The Parallel Axis Theorem states that given the rotational inertia of an object of mass M about its center of mass I_{cm}, the rotational inertia about a parallel axis through any point P, a distance h from the center of mass, is $I_P = I_{cm} + Mh^2$.
- The Perpendicular Axis Theorem states that for a planar (flat) object, the sum of the rotational inertias about two perpendicular axes in the plane of the object is equal to the rotational inertia about an axis through their point of intersection, perpendicular to them.
- Given a force applied to an object at a point not on the axis of rotation, the torque due to that force is calculated from $\vec{\tau} = \vec{r} \times \vec{F}$ where $\vec{r}$ is the position vector of the point of application of the force. In practice, the torque due to a force is found by extending the line of action of the force and dropping a perpendicular from the pivot point. This perpendicular is called the lever arm, and the torque is the force multiplied by the lever arm.
- By analogy with the linear equation $\vec{F} = m\vec{a}$, we have the angular counterpart $\vec{\tau} = I\vec{\alpha}$.
- For combined translation and rotation, the kinetic energy can be expressed as the sum of the kinetic energy of translation and that of rotation: $K = \dfrac{1}{2}Mv_{cm}^2 + \dfrac{1}{2}I_{cm}\omega^2$
- When an object rolls without slipping, the tangential velocity and acceleration of a point on the rim are equal to the linear velocity and acceleration (respectively) of the center of mass.
- For an object with momentum $\vec{p}$ at a position $\vec{r}$ from an arbitrary pivot, the angular momentum of the object about the pivot is $\vec{\ell} = \vec{r} \times \vec{p}$. In practice, angular momentum can be calculated by extending the line of action of the momentum vector $\vec{p}$ and dropping a perpendicular from the pivot (called the lever arm). The angular momentum is the linear momentum multiplied by the lever arm.
- By analogy with $\vec{p} = m\vec{v}$, $\vec{\ell} = I\vec{\omega}$.
- In general, torque is the rate of change of angular momentum: $\vec{\tau} = \dfrac{d\vec{\ell}}{dt}$. In the absence of external torques, angular momentum is conserved $\left(\dfrac{d\vec{L}}{dt} = \sum \vec{\tau}_{\text{ext}}\right)$.

AP Physics C
Unit 6 Practice Test

Name ____________________

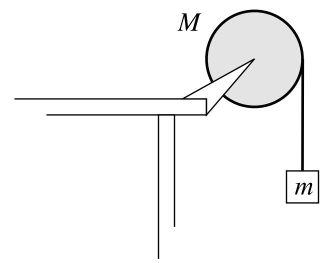
Part I
Multiple Choice

1. Two bicycle wheels with spokes of negligible mass each have the same mass of 1 kg. They start from rest and forces are applied as shown. In order to impart identical angular accelerations to the wheels, the force F_2 must be



A) 0.25 N
B) 0.5 N
C) 1 N
D) 2 N
E) 4 N

2. A uniform disk ($I = \frac{1}{2}MR^2$) of mass 8.0 kg can rotate without friction on a fixed axis. A string is wrapped around its circumference and is attached to a 6.0 kg mass. If the string does not slip on the disk, the tension in the string while the mass is falling is

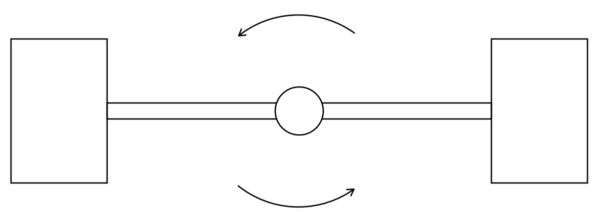


A) 20.0 N
B) 24.0 N
C) 34.3 N
D) 60.0 N
E) 80.0 N

3. The rotational inertia of a solid sphere about an axis through its center is I. The rotational inertia of another solid sphere with the same density, but twice the radius is

A) $2I$
B) $4I$
C) $8I$
D) $16I$
E) $32I$

4. A space station consists of two living modules attached to a central hub by long corridors of equal length. Each living module contains N astronauts. The mass of the space station is negligible compared to the mass of the astronauts, and the size of the central hub and living modules is negligible compared to the length of the corridors. At the beginning of the day, the space station is rotating so that the astronauts feel as if they are in a gravitational field of strength g. Two astronauts, one from each module, climb into the central hub, and the remaining astronauts now feel a gravitational field strength g'. In terms of N, the ratio $\frac{g'}{g}$ is



A) $\sqrt{\frac{2N}{N-1}}$

B) $\sqrt{\frac{N}{N-1}}$

C) $\left(\frac{N}{N-1}\right)^2$

D) $\frac{2N}{N-1}$

E) $\frac{N}{N-1}$

5. A uniform disk ($I = \frac{1}{2}MR^2$) rotates at a fixed angular velocity on an axis through its center normal to the plane of the disk, and has kinetic energy K. If the same disk rotates at the same angular velocity about an axis on the edge of the disk (still normal to the plane of the disk), its kinetic energy is

A) $\frac{1}{2}K$

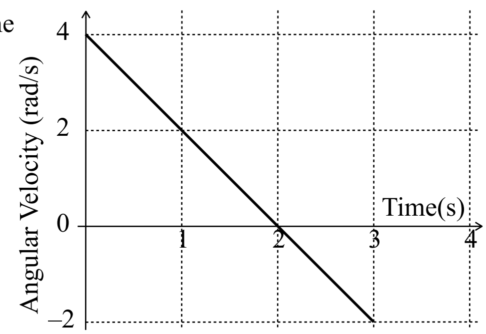
B) $\frac{3}{2}K$

C) $2K$
D) $3K$
E) $4K$

6. An energy storage device in space consists of two equal masses connected by a tether (a cord) rotating about their center of mass. Additional energy is stored by reeling in the tether; no external forces are applied. Initially the device has kinetic energy K and rotates at angular velocity ω. Energy is added until the device rotates at angular velocity 2ω. The new kinetic energy of the device is

A) $2K$
B) $4K$
C) $8K$
D) $\sqrt{2}K$
E) $2\sqrt{2}K$

For questions 7 and 8: A flat disk rotates about an axis perpendicular to the plane of the disk and through the center of the disk with an angular velocity as shown in the graph to the right.



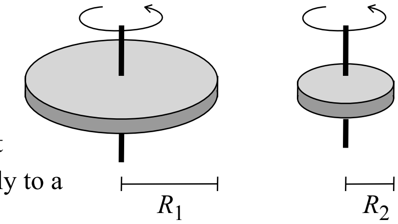
7. The angular acceleration of the disk when $t = 2.0$ s is

A) -12 rad/s^2
B) -8 rad/s^2
C) -4 rad/s^2
D) -2 rad/s^2
E) 0 rad/s^2

8. The angular displacement for the 3 second interval is

A) 9 rad
B) 8 rad
C) 6 rad
D) 4 rad
E) 3 rad

9. Two discs are mounted on thin, lightweight rods oriented through their centers and normal to the discs. These axles are constrained to be vertical at all times, and the discs can pivot frictionlessly on the rods. The discs have identical thickness and are made of the same material, but have differing radii R_1 and R_2. The discs are given angular velocities of magnitudes ω_1 and ω_2, respectively, and brought into contact at their edges. After the discs interact via friction it is found that both discs come exactly to a halt. Which of the following must hold?



A) $\omega_1^2 R_1 = \omega_2^2 R_2$

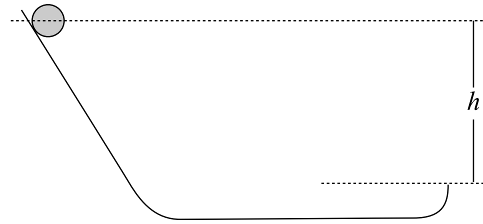
B) $\omega_1 R_1 = \omega_2 R_2$

C) $\omega_1 R_1^2 = \omega_2 R_2^2$

D) $\omega_1 R_1^3 = \omega_2 R_2^3$

E) $\omega_1 R_1^4 = \omega_2 R_2^4$

10. A ball of mass M and radius R has a rotational inertia of $I = \frac{2}{5}MR^2$. The ball is released from rest and rolls without slipping down the ramp with no frictional loss of energy. The ball is projected vertically upward off a ramp as shown in the diagram, reaching a maximum height y_{max} above the point where it leaves the ramp. In terms of h, y_{max} is



A) h

B) $\frac{25}{49}h$

C) $\frac{2}{5}h$

D) $\frac{5}{7}h$

E) $\frac{7}{5}h$

11. A hoop and a block are placed at rest at the top of an inclined plane with inclination θ above the horizontal. The hoop rolls down the plane without slipping and the block slides down the plane; it is found that both objects reach the bottom of the plane simultaneously. The coefficient of kinetic friction between the block and the plane is

A) 0

B) $\frac{1}{3}\tan\theta$

C) $\frac{1}{2}\tan\theta$

D) $\frac{2}{3}\tan\theta$

E) $\tan\theta$

12. A ball of radius 0.5 m rolls without slipping on a horizontal surface. Starting from rest at $t = 0$, the ball moves with constant angular acceleration 6.0 rad/s^2. The distance traveled by the center of the ball in 3 seconds is

A) zero
B) 13.5 m
C) 18 m
D) 27 m
E) none of these

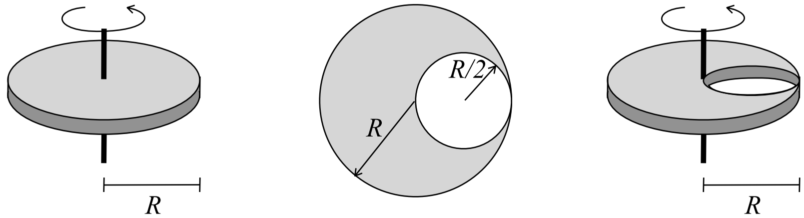
For questions 13 and 14: A figure skater has a rotational inertia of 4.0 kg•m^2 with her arms outstretched. If she begins a spin at 3.0 rad/s with her arms outstretched and then brings her arms in tight to her body, her rate of spin increases to 7.0 rad/s in 2.5 seconds.

13. As the skater brings her arms in tight to her body, the torque applied to increase her spin is

A) 2.7 N•m
B) 4.0 N•m
C) 5.3 N•m
D) 9.3 N•m
E) no torque is applied to increase her spin

14. The rotational inertia of the skater with her arms tight to her body is

A) 0.73 kg•m^2
B) 1.7 kg•m^2
C) 2.8 kg•m^2
D) 7.5 kg•m^2
E) 9.4 kg•m^2



15. A uniform disk ($I = \frac{1}{2}MR^2$) of radius R begins with a mass M. rotating about an axis through its center as shown above. A hole is cut in the disk of radius $R/2$ as shown in the second figure. In terms of the mass M and radius R of the original disk, the rotational inertia of the resulting object about the original axis is

A) $\left(\frac{15}{32}\right)MR^2$

B) $\left(\frac{13}{32}\right)MR^2$

C) $\left(\frac{3}{8}\right)MR^2$

D) $\left(\frac{9}{32}\right)MR^2$

E) $\left(\frac{5}{32}\right)MR^2$

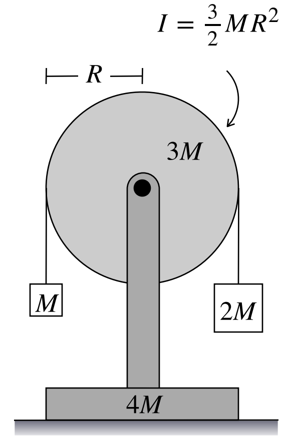
1.D 2.B 3.E 4.C 5.D 6.A 7.D 8.E 9.D 10.D 11.C 12.B 13.E 14.B 15.B

Part II

Show your work

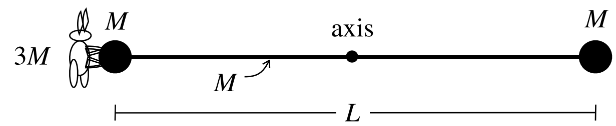
Credit depends on the quality and clarity of your explanations

1. A pulley of mass 3*M* and radius *R* is mounted on frictionless bearings and supported by a stand of mass 4*M* at rest on a table as shown to the right. The rotational inertia of this pulley about its axis is $(3/2)MR^2$. Passing over the pulley is a massless cord supporting a block of mass *M* on the left and a block of mass 2*M* on the right. The cord does not slip on the pulley, so after the block-pulley system is released from rest, the pulley begins to rotate, and the tension T_L in the cord on the left side is not the same as the tension T_R on the right.

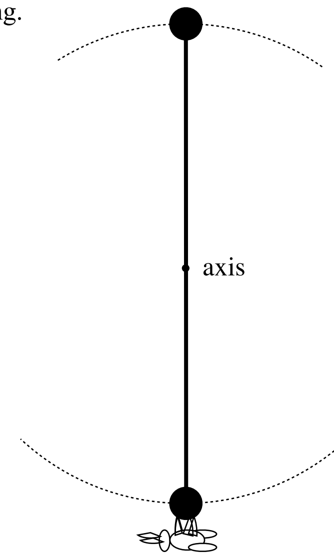


   A) Write the equations of translational motion (Newton's second law) for each of the two blocks and the analogous equation for the rotational motion of the pulley in terms of the above variables.
   B) Solve the equations in part A) for the acceleration of the blocks in terms of *g*.
   C) Determine the tensions T_L and T_R in terms of *M* and *g*.
   D) Determine the normal force exerted on the apparatus by the table when the blocks are in motion, in terms of *M* and *g*.

2. Two small spheres, each of mass M, are fastened to opposite ends of a rod of mass M and length L. This system is initially at rest with the rod horizontal, as shown to the right, and is free to rotate about a frictionless horizontal axis through the center of the rod and perpendicular to the plane of the page. A bug, of mass $3M$, lands gently on the sphere on the left. Assume that the size of the bug is small compared to L. Express your answers to the following in terms of M, L, and constants.



A) By integration, determine the rotational inertia of the rod about the axis.
B) Determine the rotational inertia of the rod-spheres-bug system.
C) Determine the net torque about the axis immediately after the bug lands on the sphere.
D) Determine the angular acceleration of the system when the rod makes an angle θ with the horizontal.
E) At the instant the rod is vertical, as shown to the right, determine each of the following.
   i) The linear speed of the bug
   ii) The angular momentum of the system

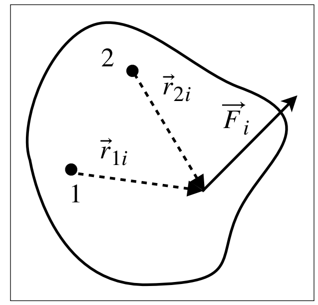


AP Physics C
Unit 7

Name____

Equilibrium

An extended object in three dimensions has 6 degrees of freedom (3 translational and 3 rotational). An extended object is said to be in *equilibrium* when there are no net forces or torques on it. We will simplify our discussion of equilibrium by confining ourselves to two-dimensional motion, so that the objects have 3 degrees of freedom (eg., translation about the *x*- and *y*-axes and rotation about the *z*-axis). For our purposes, then, equilibrium is equivalent to the three conditions: $\sum F_x = 0$, $\sum F_y = 0$ and $\sum \tau_z = 0$.



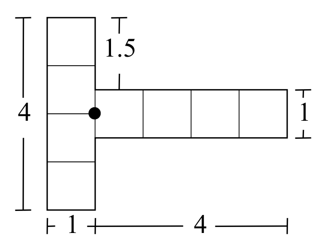
The diagram to the right represents an extended object in the plane of the page. $\vec{F}_i$ represents one (the *i*th) of many possible forces acting on the object. $\vec{r}_{1i}$ and $\vec{r}_{2i}$ represent the position vectors of the point of application of $\vec{F}_i$ with respect to arbitrary pivot points 1 and 2. Use summation notation to express the total torque due to all (*N*) such forces about each of the pivot points: $\Sigma\vec{\tau}_1 =$ [ ] and $\Sigma\vec{\tau}_2 =$ [ ] Draw and label a third vector on the diagram above that connects points 1 and 2 (in either direction). Write a vector equation that includes both $\vec{r}_{1i}$ and $\vec{r}_{2i}$ and this third vector: [ ]

Express $\Sigma\vec{\tau}_1$ in terms of this third vector and $\vec{r}_{2i}$: $\Sigma\vec{\tau}_1 =$ [ ] Use the distributive law to express $\Sigma\vec{\tau}_1$ in terms of $\Sigma\vec{\tau}_2$ and one other summation term: $\Sigma\vec{\tau}_1 =$ [ ] Explain why this summation term is equal to zero if the object is at equilibrium:

[ ]

Summarize your results by making a statement about the choice of pivot point for torques when an object is at equilibrium:

[ ]



Side view

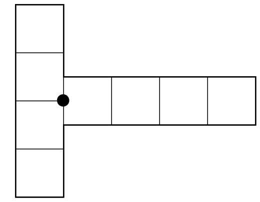
Consider the extended object in the diagram to the right, made of plywood of uniform density and dimensions in the ratios shown on the diagram. A small hole is drilled in the plywood at the point shown and the plywood is suspended on a frictionless pivot at that point. A student makes the following statement: "The plywood will remain at equilibrium because there is an equal amount of mass to the left and to the right of the pivot." Explain what is wrong with this statement.

[ ]

Using arbitrary (dimensionless) units of torque, find the magnitudes of the counterclockwise torque due to the left side of the plywood, the clockwise torque due to the right side of the plywood, and the net torque on the plywood:

$\Sigma\tau_L =$ [ ] $\Sigma\tau_R =$ [ ] $\Sigma\tau =$ [ ]

Determine the location of the center of mass of the plywood, and indicate its location on the diagram to the right. Show your reasoning:

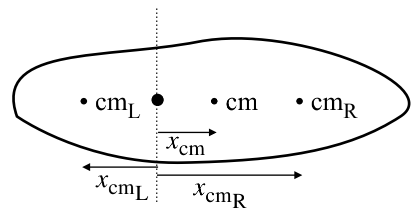


Side view

In arbitrary units, determine the torque about the pivot considering the plywood as a point at its center of mass. (Compare this with the net torque calculated on the last page.)

$\Sigma\tau =$

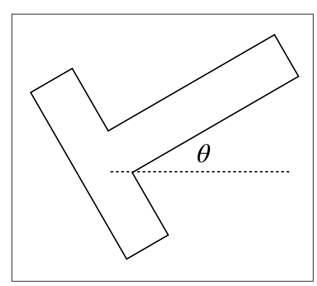
We can generalize the result you've shown above. Consider the arbitrary shape shown in the diagram to the right. The center of mass of the entire object, and the centers of mass of the left and right sections are labeled along with their *x*-coordinates with respect to the pivot. Let the mass of the entire shape be M, and the masses of the left and right sections M_L and M_R, respectively. Express the magnitudes of the gravitational torque due to the center of mass, and the torques due to the left and right sections, in terms of these masses and coordinates:



$\Sigma\tau =$ $\Sigma\tau_L =$ $\Sigma\tau_R =$

Use the definition of center of mass to show that $\Sigma\tau = \Sigma\tau_L + \Sigma\tau_R$:

The plywood from above is now suspended by a pivot through the center of mass and rotated through an angle θ as shown to the right. On the diagram, mark the center of mass (which is now the pivot). The plywood is released from rest at this position. Predict whether the plywood will initially *rotate counterclockwise*, *rotate clockwise*, or *remain at rest*, and justify your prediction.

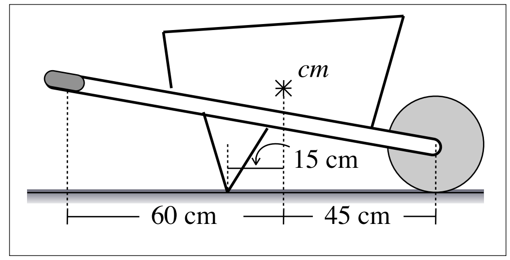


Equilibrium problems are solved using the following steps:

1. Isolate the system
2. Draw an extended free-body diagram of the system showing forces where they act
3. Choose *x* and *y* axes for the forces and a pivot for the torques
4. Apply $\sum F_x = 0$, $\sum F_y = 0$ and $\sum \tau_z = 0$

Examples

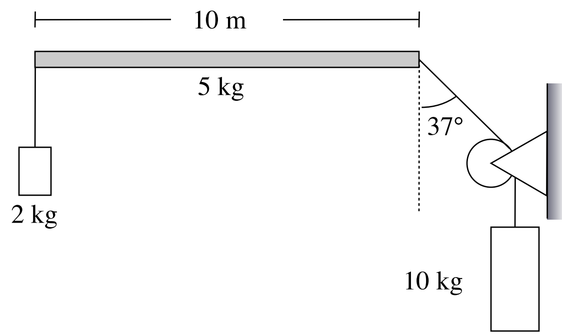
1. A wheel barrow weighs 200 N and is filled with 400 N of wet concrete. The center of mass of the wheel barrow and its contents is marked on the figure to the right.
   A) Draw vectors on the diagram to the right to show the forces acting on the wheelbarrow when it is at rest as shown.
   B) Determine the normal forces acting at the two points of contact with the ground.



   The handle is lifted vertically so that only the wheel touches the ground.
   C) What minimum force F is required to lift the handle of the wheel barrow?

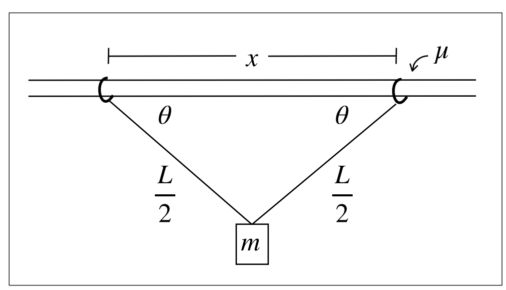
   D) How much weight does the wheel support when this force just lifts the wheel barrow off its stand?

2. A single force is to be applied to the uniform 5 kg bar shown in the figure to the right to maintain it in equilibrium in the position shown.
   A) What are the components F_x and F_y of the required force?

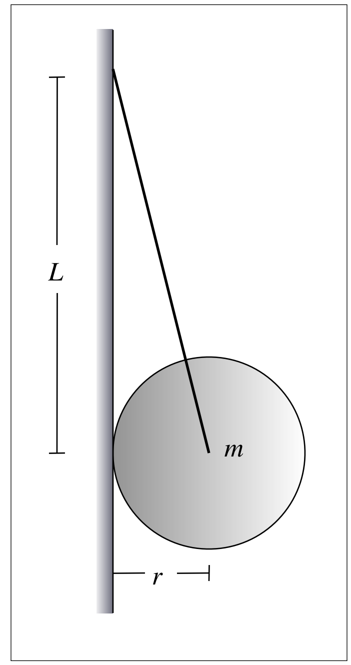


   B) What is the distance measured from the right end of the bar to the point where the force should be applied?

3. An object of mass m is attached to the midpoint of a string of length L whose ends are tied to massless rings that fit loosely around a rod as shown to the right. The coefficient of static friction between the rings and the rod is μ.
   A) Draw vectors on the diagram showing the forces acting on the left-hand ring.
   B) In terms of L and μ, find the maximum distance x between the rings such that they will not slip.



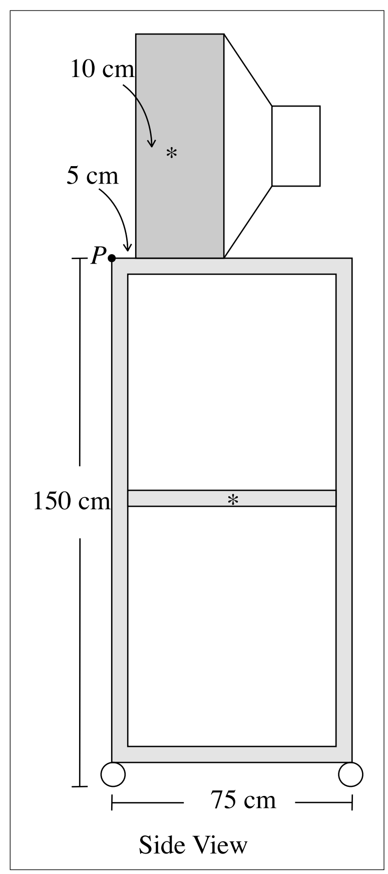
5. A uniform sphere of mass m and radius r is being held by a rope attached to a frictionless wall a distance L above the center of the sphere as shown to the right.
   A) On the figure, draw and label the forces acting on the sphere.
   B) Find the normal force exerted by the wall on the sphere in terms of m, g, r and L. Show your choice of pivot on the diagram.



   C) Find the tension in the rope (Hint: Use the Pythagorean Theorem).

   D) Check your expression for dimensional consistency, and check that it makes sense (1) when r goes to zero and (2) when r is large compared to L.

6. A television monitor weighing 200 N is strapped to the top of a rolling cart that weighs 40 N, with dimensions as shown in the diagram to the right. The center of mass of the cart is at its geometrical center, but because of the weight of the picture tube, the center of mass of the monitor is 10 cm behind the front of the screen as shown.
   A) On the diagram, draw and label vectors representing the forces acting on the cart/monitor system when it is at rest.
   B) Assume that the normal forces on the two front casters are equal and the normal forces on the two back casters are equal. Determine these normal forces.



A teacher pulls the cart toward her by applying a horizontal force to the left at point P. Due to friction between the front wheels and the floor, the cart just begins to tip; that is, the rear wheels just leave the floor.
   C) How does the free-body diagram of the the cart/monitor system differ from the one drawn in part A)?

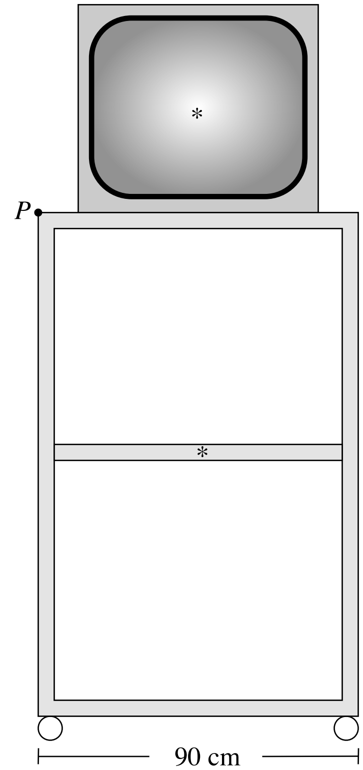
   D) Find the horizontal force at P needed to tip the cart.

E) Find the minimum coefficient of friction necessary for the cart to tip.

The teacher has learned her lesson and now pulls the cart from the side at point P, as shown to the right.

F) Find the horizontal force at P needed to tip the cart.

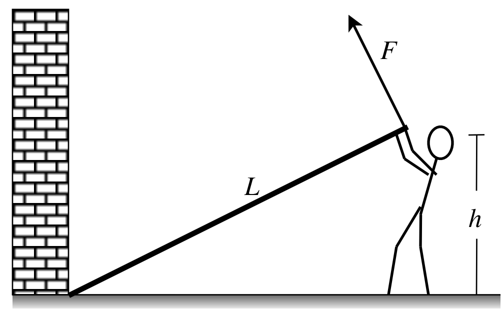
G) Find the minimum coefficient of friction necessary for the cart to tip.



Front View

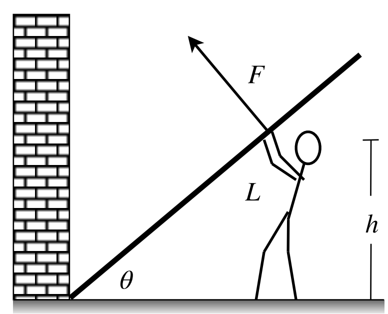
7. A man of height h picks up one end of a ladder of length L and mass m, puts the other end against a building as shown to the right.The ladder can be considered uniform, so its center of mass is at its center.

A) Find the force needed to hold the end of the ladder at the height h in terms of m, g, L and h. F is perpendicular to the ladder.



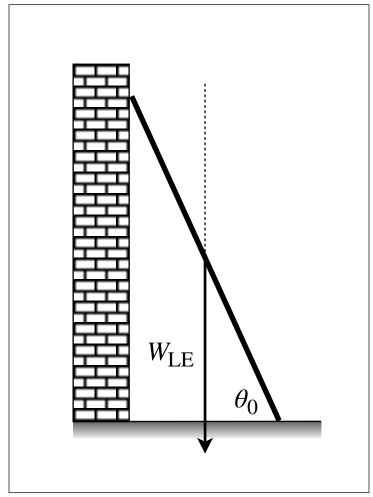
The man "walks" the ladder upright as shown in the next diagram. The force he exerts is always at the same height h and always perpendicular to the ladder.

B) As he "walks" the ladder upright, find the force needed in terms of m, g, L, h, and θ.



C) At what angle is the force he exerts a maximum?

The ladder now leans against the building at an angle θ_0 with the ground. There are small rollers at the top of the ladder, so the building can be considered frictionless. The weight vector is shown in the diagram, and its line of action is shown.



D) *Using the same scale*, draw a vector representing the normal force N_{LG} on the ladder by the ground. Explain how you know the proper scale.

E) *Using the same scale*, draw a vector representing the frictional force f_{LG} on the ladder by the ground. Explain how you know the proper scale. (Hint: By clever choice of pivot, you can determine where the vector sum of f_{LG} and N_{LG} must point so that $\Sigma\tau = 0$.)

F) *Using the same scale*, draw a vector representing the normal force N_{LB} on the ladder by the building. Explain how you know the proper scale.

G) Find the minimum coefficient of static friction μ_s such that the ladder will not slip when leaning at the angle θ_0.

Assume that the coefficient of static friction is the minimum value found in part G). The man, whose mass is 5 times that of the ladder, now wishes to climb to the top of the ladder.

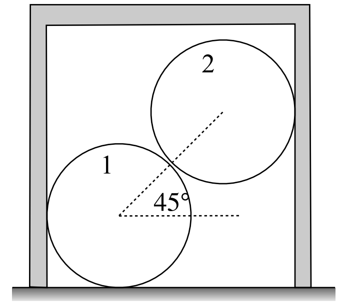
H) In terms of θ_0, find the minimum angle at which the ladder will not slip with the man at the top.

Now assume that the building exerts a frictional force f_{LB} at the top of the ladder.

I) Of the other forces drawn in the free-body diagram above, which would have a different value in this case? Explain briefly.

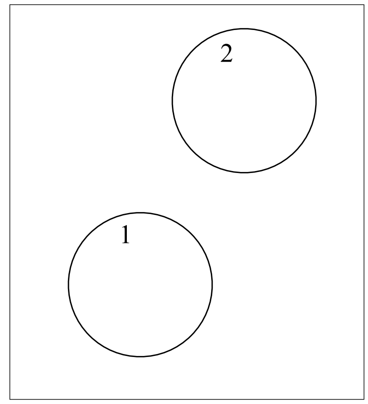
J) Choose and label a pivot, and write (but do not solve) an equation that follows from $\Sigma\tau = 0$ for this pivot.

8. Two identical, uniform, frictionless spheres 1 and 2, each of mass m and radius r, are placed on a table under a bottomless, rigid cylindrical container as shown in cross-section in the figure. The diameter of the container is $\left(2+\sqrt{2}\right)r$, so the line joining centers of the spheres makes a 45° angle with the horizontal.



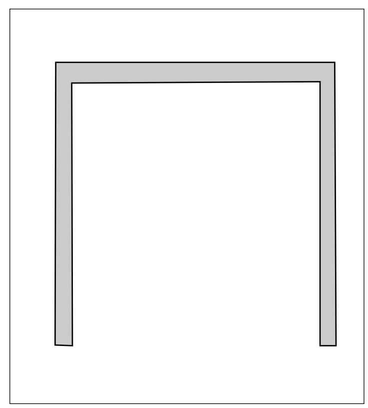
A) In the space below, right, draw and label the forces acting on each sphere, using *on* subscripts.

B) Determine the magnitude of each of the forces on the spheres in terms of m and g.

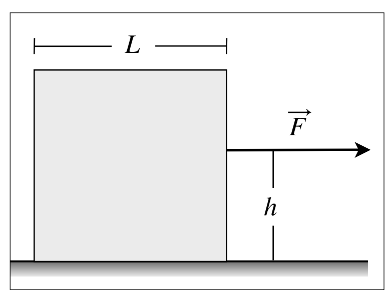


C) In the space provided, draw and label the forces acting on the container, using *on* and *by* subscripts.

D) Find, in terms of m, the minimum mass of the container such that it will not start to tip.



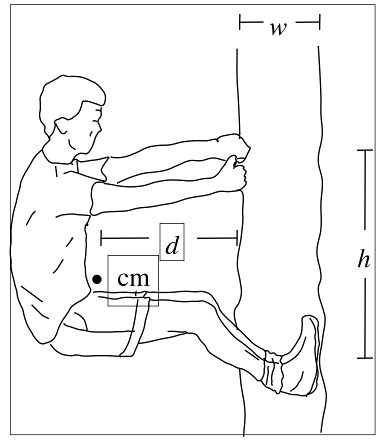
9. A uniform cube of side L is on a horizontal floor with a coefficient of static friction μ. A horizontal force $\vec{F}$ is applied at a distance h above the floor on the midline of one of the cube faces. As $\vec{F}$ is slowly increased, the cube will either slide or begin to tip, depending on the value of μ relative to L and h.



A) In the space to the right, draw and label the forces acting on the cube if it is about to tip.

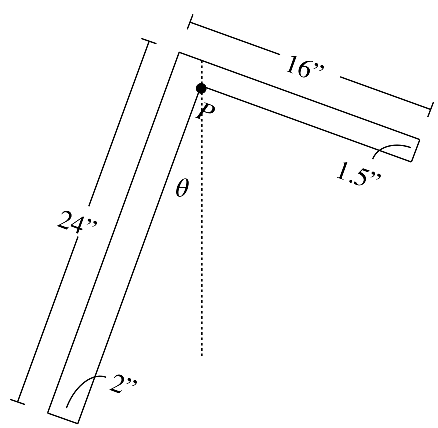
B) Determine the minimum value of μ in terms of L and h such that the cube will tip before it slides.

10. A 55 kg. rock climber is in a lie-back climb along a fissure, with hands pulling on one side of the fissure and feet pressing against the opposite side. The fissure has width $w = 0.2$ m, and the center of mass of the climber is a horizontal distance $d = 0.4$ m from the fissure. The coefficient of friction between hands and rock is 0.4, and between boots and rock is 1.2.
   A) On the diagram, draw vectors representing the forces acting on the rock climber, using *on* and *by* subscripts.
   B) What is the least horizontal pull by the hands and push by the feet that will keep him stable?

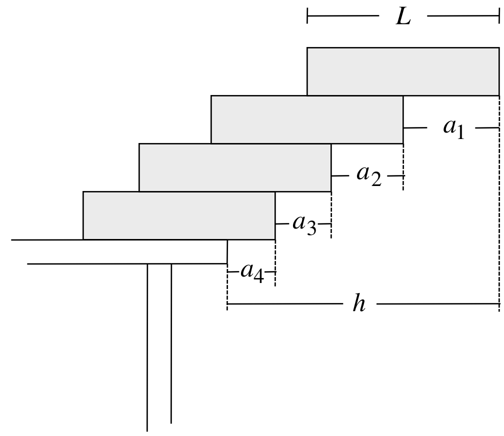


   C) For the horizontal pull of part B), what must be the vertical distance h between hands and feet?

11. A carpenter's square is made of 1/16" thick steel in the dimensions shown to the right. It is hung on a nail at point P. Find the angle that the long side makes with the vertical when it hangs. (Hint: When it hangs at equilibrium, the center of mass is vertically under point P.)



12. Four bricks of length L, identical and uniform, are stacked on top of one another in such a way that part of each extends beyond the one beneath.



A) Find, in terms of L, the maximum value of a_1 such that the top brick is in equilibrium. Explain your reasoning.

B) Assuming that a_1 is the maximum value found above, and find, in terms of L, the maximum value of a_2 such that the top two bricks are in equilibrium. Explain your reasoning.

C) Similarly, find, in terms of L, the maximum values of a_3 and a_4 such that all four bricks are in equilibrium.

D) Find the maximum value of h such that the stack is in equilibrium.

E) Express using summation notation the value of h for a total of N bricks stacked in this way.

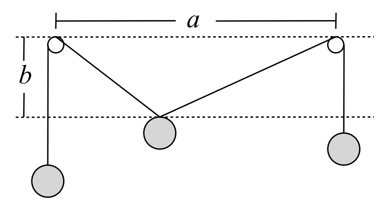
F) With an unlimited number of bricks stacked in this way, what is the maximum possible value of h?

AP Physics C
Unit 7 Practice Quiz

Name ___________________________________

Part I
Multiple Choice

1. The three masses shown to the right are equal. The pulleys are small, and friction is negligible. Assuming that the system is at equilibrium, what is the ratio a/b? (Note: The figure is not drawn to scale.)



A) 1/2
B) 1
C) $\sqrt{3}$
D) 2
E) $2\sqrt{3}$

2. Lucy (mass 33.1 kg), Henry (mass 63.7 kg) and Mary (mass 24.3 kg) sit on a lightweight seesaw at evenly spaced 2.74 m intervals, in the order in which they are listed. If the seesaw balances, who exerts the torque with the greatest magnitude?

A) Henry
B) Lucy
C) Mary
D) They all exert the same torque
E) Lucy and Mary exert the same torque, which is greater than Henry's

3. A meter stick is supported at each end by a spring scale. A heavy mass is then hung on the meter stick so that the spring scale on the left reads four times the value of the spring scale on the right. Neglecting the mass of the meter stick, how far from the right end of the meter stick is the heavy mass hanging?

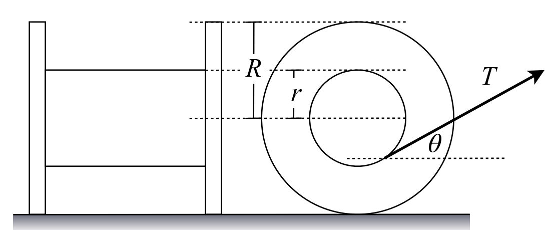
A) 25 cm
B) 50 cm
C) 67 cm
D) 75 cm
E) 80 cm

4. Three forces act on an object. If the object is in translational equilibrium, which of the following must be true?

I The vector sum of the three forces must equal zero
II The magnitudes of the three forces must be equal
III All three forces must be parallel

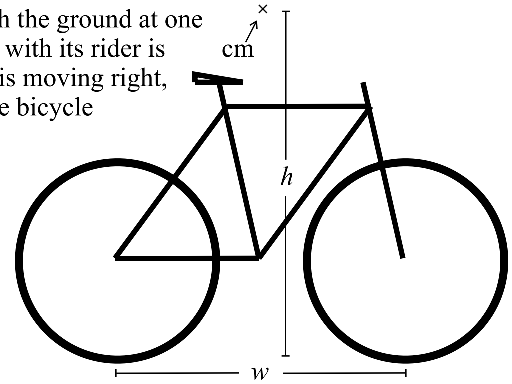
A) I only B) II only C) I and III only D) II and III only E) I, II, and III

5. A string is wrapped around a spool of weight W with inner radius r and outer radius R as shown to the right. The string is pulled with a tension T and an angle θ with the horizontal. As a result, the spool slides at a constant speed without rotating. Which of the following conditions must hold?



A) $T = W$
B) $T = W \sin\theta$
C) $T = W \cos\theta$
D) $\cos\theta = \dfrac{r}{R}$
E) $\sin\theta = \dfrac{r}{R}$

For questions 6 and 7: A bicycle has two tires that each come into contact with the ground at one point. The wheelbase of the bicycle is w, and the center of mass of the bicycle with its rider is located midway between the tires at a height h above the ground. The bicycle is moving right, but the brakes are applied so that the tires are skidding without rotating, and the bicycle is slowing down at a constant rate with acceleration a. The coefficient of kinetic friction between the tires and the ground is μ.



6. The maximum value of μ such that both tires remain in contact with the ground is

   A) $\dfrac{2h}{w}$

   B) $\dfrac{h}{2w}$

   C) $\dfrac{w}{2h}$

   D) $\dfrac{w}{h}$

   E) None of these

7. The maximum value of a such that both tires remain in contact with the ground is

   A) $\dfrac{2wg}{h}$

   B) $\dfrac{wg}{h}$

   C) $\dfrac{hg}{2w}$

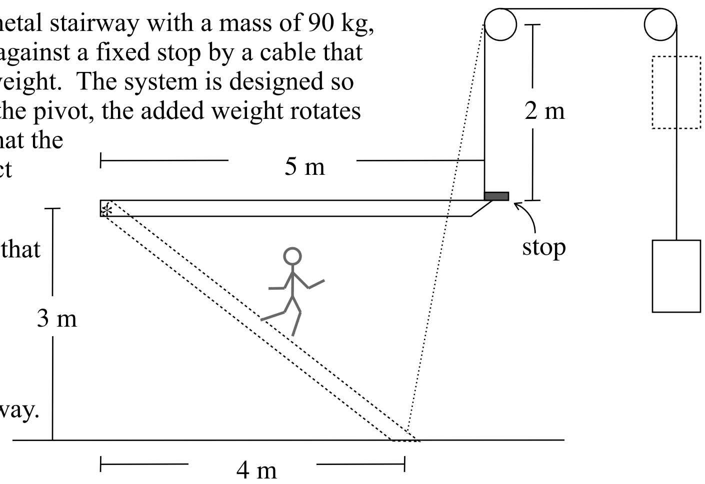
   D) $\dfrac{h}{2wg}$

   E) None of these

1. E 2. B 3. E 4. A 5. D 6. C 7. E

Part II
Show your work
Credit depends on the quality and clarity of your explanations

1. The bottom of a fire escape consists of a 5 meter long metal stairway with a mass of 90 kg, pivoted at the left end and held in a horizontal position against a fixed stop by a cable that runs over two pulleys as shown and down to a counterweight. The system is designed so that when a person steps on the stairway to the right of the pivot, the added weight rotates the stairway down to the ground, 3 m below. Assume that the center of mass of the stairway is at its center, and neglect friction.



A) Find the minimum mass of the counterweight such that the empty stairway will be held against the stop.

Assume that the mass of the counterweight is 10 kg greater than the value found in part A).

B) A person with a mass of 45 kg walks onto the stairway. What distance, measured from the pivot, must she walk before the stairway starts to swing down?

C) Find the distance, measured from the pivot, that the person must walk along the stairway in order that it swing down to the ground.

AP Physics C
Unit 8 Name________________________

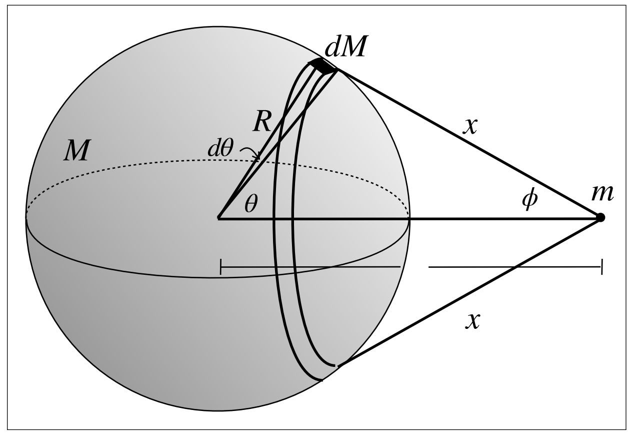
Newton's Law of Universal Gravitation

In 1665 Newton theorized that there is an attractive force between any two particles of matter in the universe, whose strength is proportional to the product of the two masses, and inversely proportional to the square of the distance between them. If m_1 and m_2 are the two masses and r is the distance between them (r is usually *not* the radius of a circle), then the law can be written

$$F = \frac{G m_1 m_2}{r^2}$$

where G is a proportionality constant. In SI units, $G = 6.67 \times 10^{-11}\ \frac{\text{Nm}^2}{\text{kg}^2}$.

Extended objects such as human beings, planets and stars exert gravitational forces on each other by virtue of being composed of particles. In general, one must integrate over all the particles in each body to find the total gravitational force. This is usually very complicated except in the case of objects with *spherical symmetry*. For those, as Newton guessed, the net gravitational effect on other objects is the same as that of a point particle with the same mass at the center of the sphere. Newton went on to prove his guess, inventing integral calculus partly for this reason. You will solve the problem below using modern methods.



A spherically symmetric object can be thought of as made of layers of thin, concentric uniform spherical shells. Consider the spherical shell of mass M and radius R drawn to the right. A point mass m is at a distance r from the center of the shell. A section of the shell labeled dM is shown, at a distance x from the point mass. According to Newton's Law, what is the force of attraction on m? [ ] Draw this force on the diagram to the right and label it dF_0.

Now imagine that dM is spread out over the ring in the drawing, where each point on the ring is the same distance x from the point mass. For this new dM, the forces of attraction on m would form what shape? [ ] Sketch this shape on the diagram. In what direction would the vector sum of these forces point? [ ] Draw the vector representing this net force and label it dF.

What is the magnitude of this force? [ ] To find the force on the point mass due to the entire shell, we have to integrate the forces due to all such rings. Given the parameters R, r, x, θ and ϕ labeled on the diagram, which will vary as we integrate over the shell? [ ] We need to express dF in terms of only one of these.

Find the ratio $\frac{dM}{M}$ and simplify: $\frac{dM}{M} =$ [ ]

Express $\cos\phi$ in terms of r, R, x and θ: $\cos\phi =$ [ ]

Apply the Law of Cosines to get an expression involving x, r, R and $\cos\theta$:

Solve for $\cos\theta$: $\cos\theta =$

Differentiate both sides to find an expression for $\sin\theta\, d\theta$:

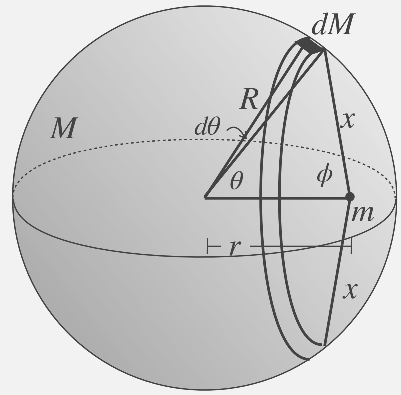
$\sin\theta\, d\theta =$

Now rewrite the expression for dF eliminating all variables except x and dx. Simplify by factoring all constants out:

Integrate the expression for dF using appropriate limits:

State in your own words what the result of this integration means:

The diagram to the right shows the same parameters for a point mass moved *inside* the shell at a distance $r < R$ from the center of the shell. Review the steps in the derivation above and write the appropriate definite integral you can evaluate to find the net force on this point mass:



Integrate with your chosen limits to find the net force on m:

State in your own words what the result of this integration means:

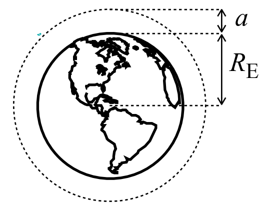
Explain why the results of your integrations can be extended to apply to solid spheres. Do the solid spheres have to have uniform density? Explain why or why not.

We are now justified in claiming that the gravitational force acting on objects at the surface of a planet such as Earth is the same as if the planet's mass were concentrated at its center. We have used the term "weight" (mg) to describe this force, where "g" is the acceleration due to gravity at the surface. If the planet has mass M and radius R, derive an expression for g at the surface:

More generally, we can let g represent the *gravitational field strength* at any point, whether on the surface of a planet or not.

Examples

1. Consider a satellite in low earth orbit, at an altitude a above the surface of the earth.

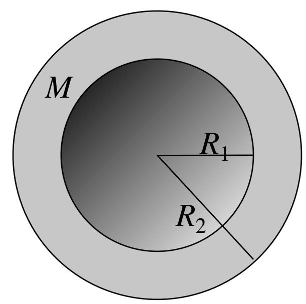


   A) Derive an expression for the gravitational field strength g_a at this altitude in terms of g at the surface, the altitude a and the radius of the earth R_E

   B) The radius of the earth is about 6.37×10^6 m and a typical altitude for low earth orbit is about 300 km. Use your expression to find the strength of the gravitational field at this altitude.

   C) In terms of R_E, at what altitude from the surface of the earth would g_a be half of its value at the surface?

2. Consider a planet in the form of a thick spherical shell as shown in the diagram to the right, with mass M, inner radius R_1 and outer radius R_2. Write expressions for the gravitational field strength at the following locations:

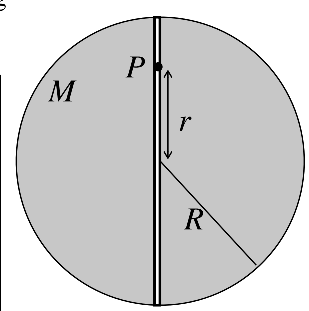


   A) $r < R_1$

   B) $r > R_2$

   C) $R_1 < r < R_2$:

3. A planet has uniform density, mass M and radius R. A hole is drilled through the planet along a diameter. Find an expression for the gravitational field strength g_r at point P, a distance $r < R$ from the center of the planet.



Gravitational Potential Energy

Recall that gravitational potential energy U_g is the work needed to move a given object to its current position from some arbitrary zero-energy position. Near the surface of the earth, where the gravitational field is fairly uniform, the force is constant (mg), so the work is mgh, where h is the altitude above (or below) the arbitrary zero level. On a planetary scale, the gravitational force is not uniform, we need to reformulate our definition of gravitational potential energy.

Consider a small mass m moving in the gravitational field of a planet of mass M. The force on m (equal and opposite to the force on M) depends only on the separation distance r, so it is a conservative force. With every conservative force F there is a potential energy function U which represents the work needed to move the system from the configuration where $U = 0$.

There is some gravitational force at *any* finite separation distance, so it is convenient to choose infinite separation distance as the $U = 0$ position, where the gravitational force is zero. Since gravity is an attractive force, this means that every finite separation distance will have a *negative* potential energy.

To find the potential energy at any separation distance r, we find the work needed to change from infinite separation to a separation of r. Write the definite integral that expresses this statement:

Evaluate the integral to determine the (negative) potential energy of the system at a separation distance r:

$U_g =$

Note that U_g is a property of the *system* of two masses; one mass alone has no U_g because it effectively has infinite separation from all other masses. Since energy is a scalar, the gravitational potential energy of a system of three or more masses is the sum of the energies calculated pairwise.

Example

Use the data in the table to the right to calculate U_g for the earth-moon-sun system.

Earth mass: $M_E = 5.98 \times 10^{24}$ kg
Sun mass: $M_S = 1.99 \times 10^{30}$ kg
Moon mass: $M_M = 7.36 \times 10^{22}$ kg
$r_{\text{Earth-Moon}} = 3.82 \times 10^{8}$ m
$r_{\text{Earth/Moon-Sun}} = 1.5 \times 10^{11}$ m

Escape Velocity

On a planetary scale, a small mass m at rest on the surface of a planet of mass M and radius R has negative potential energy.

Write the expression for this energy: $U_g =$ How much (positive) kinetic energy would this mass have to be given at the surface of the planet to escape to infinity with no energy left over?

Determine the velocity the mass would need at the surface of the planet to escape:

This is called the *escape velocity*.

Examples

1. Use the data on the previous pages to calculate v_{esc} for the earth.

2. To create a planet with a large escape velocity, you need to give it a large mass and pack it into a small radius. If the escape velocity is greater than or equal to the speed of light ($c = 3 \times 10^8$ m/s), then nothing (not even light) could escape from it. Such an object is called a *black hole*. For a given object, the *Schwarzchild radius* is the radius of the sphere into which the object would have to be packed to make a black hole.
   A) Derive an expression for the Schwarzchild radius of an object of mass M. Evaluate the proportionality constant.

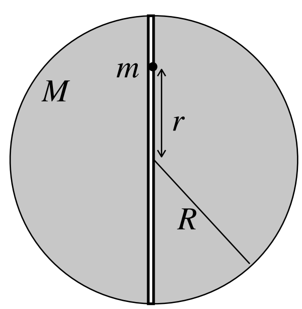
   B) Use the data in the table on the previous page to calculate the Schwarzchild radius of the earth.

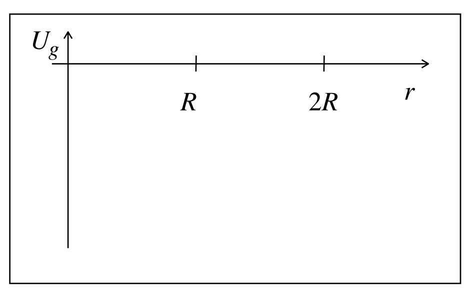
   C) Use the data in the table on the previous page to calculate the Schwarzchild radius of the sun.

   D) Calculate the Schwarzchild radius of a galaxy with one trillion (10^{12}) stars each with the mass of the sun.

   E) Imagine a planet with a mass equal to one trillion suns and a radius equal to the Schwarzchild radius. Calculate the gravitational field strength at the surface. Is this *much greater than*, *much less than*, or *about the same as* at the surface of the earth?

3. A planet like the one on page 8.4 has a hole drilled along a diameter. A small mass m moves along the hole from the surface to the center of the planet. Determine the gravitational potential energy of the planet-mass system when the mass is at the center of the planet, and sketch a graph of the gravitational potential energy when the mass is both inside and outside the planet on the axes to the right.





Orbits and Kepler's Laws

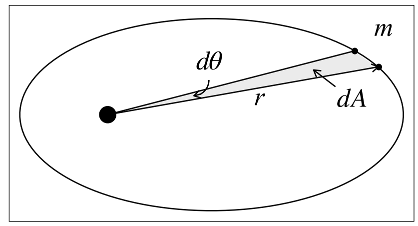
Around 1610, before the use of the telescope for astronomy, Johannes Kepler stated three laws which seemed to summarize the available observations of the planets and their motion around the sun. Later Newton was able to prove the three laws based on his law of gravitation, and assuming that the sun is so large that is essentially fixed in space. Kepler's laws are:

1. The planets move in elliptical paths with the sun at one focus.
2. A line from any planet to the sun sweeps out equal areas in equal times.
3. If T is the period of a planet's orbit, and R the average distance from the sun, then R^3/T^2 is the same for every planet.

These laws hold not only for the solar system, but any "planetary system" consisting of one very large central attracting object that's considered fixed in space.

Kepler's First Law is difficult to show, but follows from the $1/r^2$ nature of the gravitational force. More generally, Newton was able to show that for this kind of force law, the planet would follow one of three paths depending on its total energy:

- If the total energy is negative, the planet would move on an ellipse (a circle is a special case of an ellipse).
- If the total energy is zero, the planet would move on a parabola.
- If the total energy is positive, the planet would move on a hyperbola.

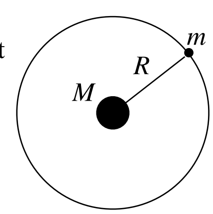


Kepler's Second Law is known as the *area law*. Consider a planet of mass m in an elliptical orbit around the sun, as shown to the right. In a short time dt it sweeps through an angle $d\theta$, and sweeps out an area dA. The claim of the area law is that $\dfrac{dA}{dt}$ is constant.

The area dA is approximately a triangle; write an expression for dA in terms of r and $d\theta$: $dA =$ Use this to express the ratio $\dfrac{dA}{dt}$: $\dfrac{dA}{dt} =$ Express the ratio in terms of the angular velocity ω of the planet: $\dfrac{dA}{dt} =$ On the diagram above, draw a vector to represent the force on the planet in the two positions shown. Is the linear momentum of the planet a constant? Explain why or why not.

Is the angular momentum of the planet about the sun a constant? Explain why or why not.

Explain why $\dfrac{dA}{dt}$ is constant under these conditions.



Kepler's Third Law can be proven for circular orbits as follows: The diagram to the right shows a planet of mass m in a circular orbit around a much more massive central object of mass M. What is the magnitude of the gravitational force on m? Apply Newton's Second Law to the planet, expressing the centripetal acceleration in terms of the period *T*. Solve this

expression for $\frac{R^3}{T^2}$: $\frac{R^3}{T^2} =$ Explain why this proves Kepler's Third Law for circular orbits.

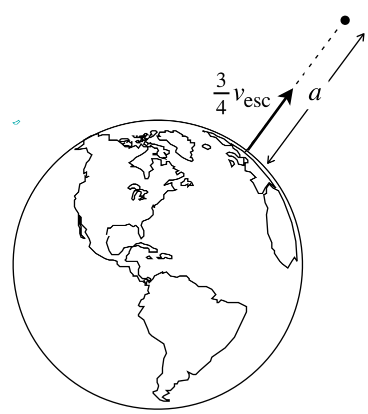
Examples

1. The planet Neptune is about 30 times as far from the sun as Earth. Determine Neptune's orbital period in Earth-years.

2. Use Kepler's Third Law to find the period of a satellite in low-earth orbit at an altitude of 300 km (see page 8.4).

3. An object of mass *m* is projected vertically from the earth's surface with an initial speed 3/4 of the escape velocity. Answer the following in terms of *m*, Earth's mass *M* and radius *R* and constants:

   A) Determine its maximum altitude *a* reached by the object.

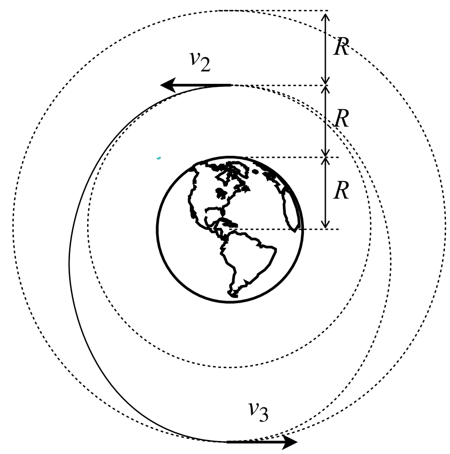


   B) Determine the change in gravitational potential energy ΔU_g as the object goes from the surface to an altitude *R* above the surface (2*R* from the center) .

   C) Determine the speed of the object at an altitude *R* above the surface.

D) Determine the additional energy required to insert the object into a circular orbit at an altitude R.

In order to move the object from this circular orbit to a circular orbit at an altitude $2R$ ($3R$ from the center), it must first be boosted into a transfer orbit, an ellipse with a closest approach (perigee) of $2R$ and a farthest distance (apogee) of $3R$, measured from the center of the earth. It is then boosted again into a circular orbit.



E) For the transfer orbit, the object's speed must be increased to a speed v_2 at perigee, and it will have a speed v_3 at apogee. Use conservation of angular momentum to determine v_3 in terms of v_2.

F) Use conservation of energy to determine v_2.

G) Determine the energy boost needed to place the object into the transfer orbit from the original circular orbit.

H) Determine the energy boost needed to place the object into the new circular orbit from the transfer orbit.

4. NASA plans a mission to place a satellite into a circular orbit around the planet Jupiter (mass M_J).

   A) If the radius of the planned orbit is R, find the following in terms of R, M_J and constants:

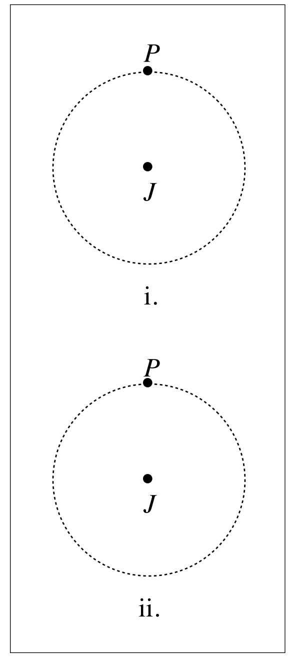
      i. The orbital speed of the planned satellite

      ii. The period of the orbit

   B) NASA wants the satellite's orbit to be synchronized with Jupiter's rotation. This requires an equatorial orbit whose period equals Jupiter's equatorial rotation period of 9 hr 51 min = 3.55×10^4 s. Given Jupiter's mass of $M_J = 1.90 \times 10^{27}$ kg, determine the required orbital radius in meters.

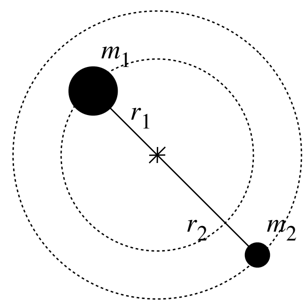
C) Suppose that the injection of the satellite into orbit is less than perfect. For an injection velocity that differs from the desired value in each of the following ways, sketch the resulting orbit on the figure. (*J* is the center of Jupiter, the dashed circle is the desired orbit, and *P* is the injection point.) Describe the resulting orbit qualitatively but specifically, and explain your reasoning.

i. When the satellite is at the desired altitude over the equator, its velocity vector has the correct direction, but the speed is slightly <u>faster</u> than the correct speed for a circular orbit of that radius.



ii. When the satellite is at the desired altitude over the equator, its velocity vector has the correct direction, but the speed is slightly <u>slower</u> than the correct speed for a circular orbit of that radius.

5. **Binary Stars** Many of the stars we see at night are *binary stars*, consisting of two stars of comparable size orbiting about their common center of mass with a common period. Consider two masses m_1 and m_2, orbiting about their center of mass as shown to the right. The distance between the masses is *not* the same as the radius of the orbit, as it was for a planetary system.



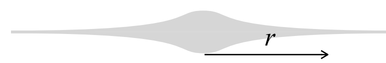
A) Determine the gravitational potential energy of the binary star system.

B) Determine the period of the binary star system.

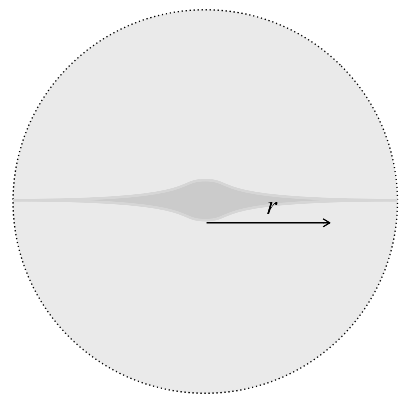
C) Determine the speeds v_1 and v_2 of each of the stars.

D) Determine the total mechanical energy of the binary star system.

6. **Dark Matter** A typical spiral galaxy consists of a massive central core surrounded by relatively thin arms of stars and gas, as shown in the edge-on view to the right.



A) Neglecting the gravitational effect of the arms' stars on each other, and considering only their attraction to the massive central core, how would you expect the orbital velocities v of the arms' stars to vary with their distance r from the central core?



B) If the galaxy is surrounded by a spherical "halo" of mass of uniform density, such that the mass of the halo is much larger than the mass of the galaxy, how would you expect the orbital velocities v of the arms' stars to vary with their distance r from the central core?

C) If the material in the halo has a density $\rho(r)$ that is not uniform, but varies as $\rho(r) = \dfrac{C}{r^2}$, where C is a constant,

ii) Write an expression for the mass contained in a thin spherical shell of radius r and thickness dr within the halo.

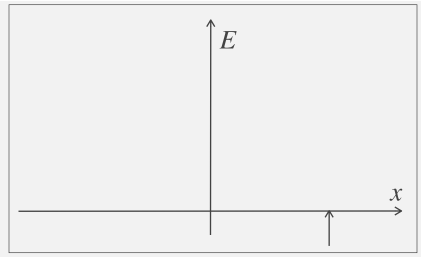
ii) Integrate to determine the mass M_{in} inside a sphere of radius r within the halo.

iii) How would the orbital velocities of the stars vary with r in this case?

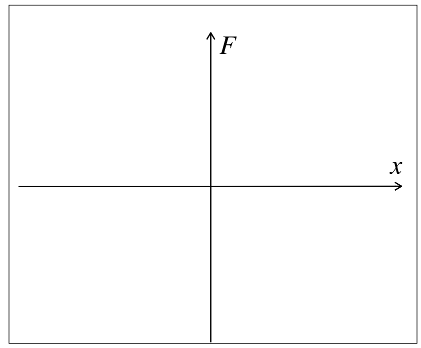
Observations of the orbital speeds confirm that they are fairly constant, almost independent of r. The material in the halo is called *dark matter*.

Simple Harmonic Motion (SHM)

Consider a one-dimensional conservative force; ie., a force that is a function of position $F(x)$. With every such force there is an associated potential energy function $U(x)$. On the axes to the right, sketch a potential energy function $U(x)$ that has one point of *stable equilibrium* located at $x = 0$. Explain why $x = 0$ is a point of stable equilibrium.

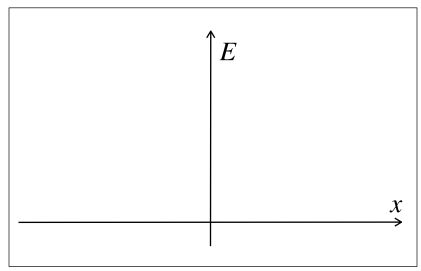


Given a particle placed at rest at the position shown by the arrow on the graph, describe how it would move under the influence of your potential energy function.



On the axes to the right, sketch a graph of the *simplest* force that would have a point of stable equilibrium at $x = 0$. Such a force is called a *restoring force*. Explain why.

Write the equation for this function. If you need a constant, call it k. (This function is known as *Hooke's Law.*)



What is the potential energy function associated with this force function?

Sketch its graph on the E vs. x axes to the right, taking the origin as the arbitrary zero of potential energy.

The Position Function for SHM

Use Newton's Second Law to write Hooke's Law in terms of x and its second derivative with respect to time. This is called a *second order differential equation*.

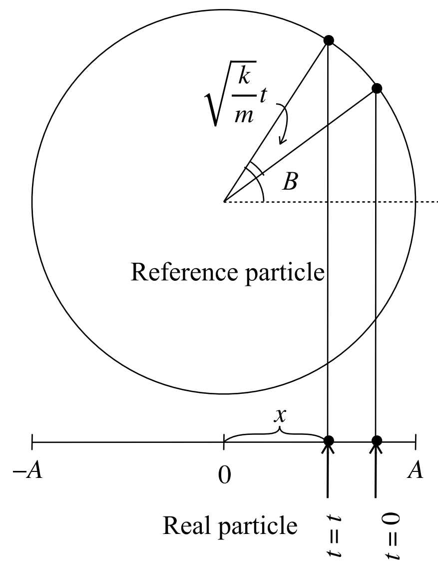
A general solution of this differential equation is of the form $x = A\cos\left(\sqrt{\frac{k}{m}}t + B\right)$, where A and B are arbitrary constants. By differentiating twice, show that this is a solution to the differential equation no matter what A and B are.

Both of the constants A and B give us information about the motion of the particle. Given that the cosine function is bounded by +1 and –1, what does the constant A tell us about the motion of the particle?

The constant A is called the *amplitude*.

The *period* (T) of periodic motion is the time it takes for the particle to move through one complete cycle of its motion. The *frequency* (f) is the reciprocal of the period ($f = \frac{1}{T}$); it is the number of cycles per unit time. (Some books use the symbol ν (the Greek letter nu, pronounced "nyew") for frequency.) Use your knowledge of the cosine function, and the equation above for $x(t)$ to express the period T and frequency f in terms of m and k.

The Reference Circle



To visualize the meaning of the equation for x as a function of t, we consider an imaginary particle moving on a circle whose radius is equal to the amplitude A of the SHM of the real particle. The particle moves around the circle in such a way that the angle it makes with the horizontal is $\sqrt{\frac{k}{m}}t + B$, as shown to the right.

Examine the diagram, and write the equation for x:

Consider two particles moving in SHM with the same amplitude and period; one with $B = 0$, and one with $B = \pi$. Describe how the motion of these two particles would be the same, and how the motion would be different.

Describe in your own words what the constant B tells us about the motion of the real particle.

Particles that move in SHM with the same amplitude and period, but lag behind or lead one another, are said to have different "phase." The constant B is called the *phase constant* or *phase angle*, and it is symbolized by the Greek letter ϕ (phi). The angular position of the reference particle can then be written $\sqrt{\frac{k}{m}}t + \phi$.

From rotational kinematics, write the equation for the angular position as a function of time when angular acceleration is zero: $\theta =$ In the expression for the angular position of the reference particle, what corresponds to angular velocity ω? For this reason, the constant $\sqrt{\frac{k}{m}}$ is called the *angular frequency* and is symbolized by ω. So the position function for SHM is:

$$x = A\cos(\omega t + \phi)$$

Derive the equation for the velocity of a particle in SHM: $v =$ What is the maximum velocity? At what point in its motion does the particle have this velocity?

Derive the equation for the acceleration of a particle in SHM: $a =$ What is the maximum acceleration? At what point in its motion does the particle have this acceleration?

Express the acceleration as a function of x:

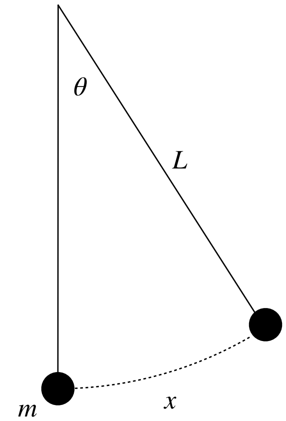
Show how your expressions for x and a satisfy both Hooke's Law and Newton's Second Law:

Energy in SHM

Use your expression for potential energy in SHM (page 8.13) to write the expression for the potential energy as a function of *time*: Use your expression for the velocity above to write the expression for the kinetic energy as a function of *time*:

Show that the sum of the potential and kinetic energies is constant. Express the constant total energy (a) as a function of A and (b) as a function of the maximum velocity v_{max}.

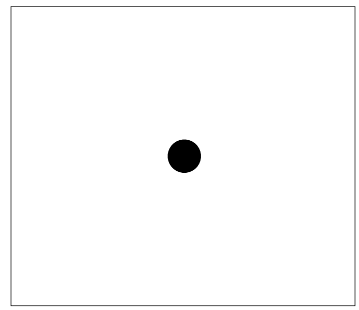
Examples of SHM



1. In order to show that an object moves in simple harmonic motion, it is sufficient to show that there is a restoring force that is proportional to its displacement from equilibrium. As an example, consider a simple pendulum consisting of a small mass m on the end of a string of length L. The motion of a pendulum is simple harmonic only as an approximation, for small displacement angles.

   A) On the diagram below, draw a free-body diagram of the mass when it is at an angle θ from the vertical. Choose axes parallel and perpendicular to the string and resolve forces along those axes.

   The object is to show that the force that tends to bring the object back toward equilibrium is proportional to the displacement x. When θ is small, $\sin\theta$ can be approximated by θ, since $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$.

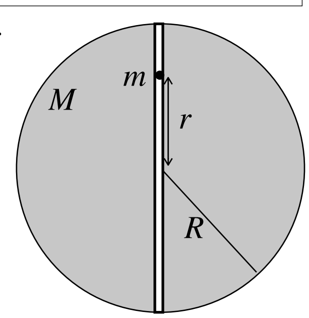
   

   B) Using this approximation and the definition of radian measure of an angle, express the component of force along the arc length x as a function of x, and show that it satisfies Hooke's Law.

   C) What is the proportionality constant k for Hooke's Law in the case of a simple pendulum? What is the period of a pendulum?

   When the amplitude of oscillation of a simple pendulum is *not* small, the period can be shown to be $T = 2\pi\sqrt{\dfrac{L}{g}}\left(1 + \dfrac{1}{2^2}\sin^2\dfrac{\theta_m}{2} + \dfrac{1}{2^2}\dfrac{3^2}{4^2}\sin^4\dfrac{\theta_m}{2} + \ldots\right)$, where θ_m is the maximum angular displacement.

   D) Using just the first two terms of the series, determine the value in parentheses when $\theta_m = 90°$. What is the percent error involved in using the small angle formula for this θ_m?



2. Consider the planet of mass M and radius R (page 8.4) with a hole drilled through a diameter. A mass m is dropped into the tunnel. In the diagram to the right, the mass is shown at a distance r from the center.

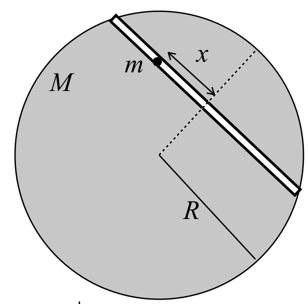
   A) What is the force of attraction on m toward the center of the planet?

B) Show that in the absence of friction the mass will go in simple harmonic motion, and determine the period.

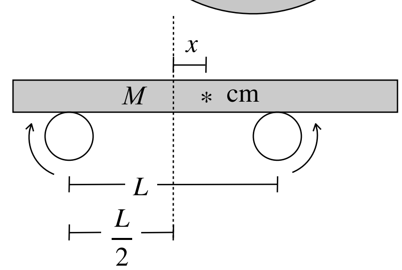
C) Using data from earlier in the packet, estimate the period and the maximum speed for such a tunnel through the earth.

D) Determine the gravitational potential energy of the planet-mass system when the mass is at the center of the planet.

E) If the tunnel were cut through the earth along a chord (ie., not a diameter), and friction were again ignored, show that the mass would still move in SHM with the same period.

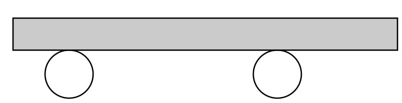


3. Two parallel cylinders of equal dimensions are turned by a motor in opposite directions as shown in the diagram to the right. Onto the cylinders is placed horizontally a uniform plank of mass *M*, such that its center of mass is a distance *x* from the point midway between the two cylinders. The distance between the cylinder axes is *L* and the coefficient of kinetic friction between the cylinders and the plank is μ.



A) In the space to the right, draw and label each of the forces acting on the plank.

B) Of the three conditions for equilibrium; $\sum F_x = 0$, $\sum F_y = 0$ and $\sum \tau = 0$, which are true for the plank?

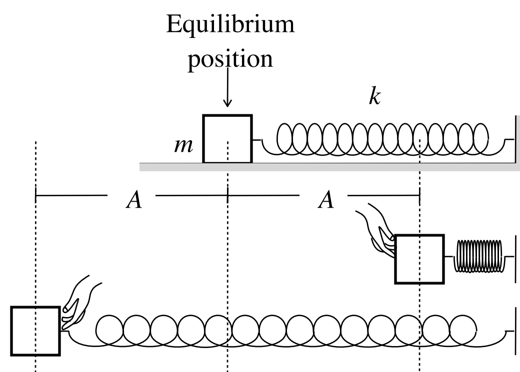


C) Determine the two normal forces in terms of the given quantities. Show your choice of pivot on the diagram.

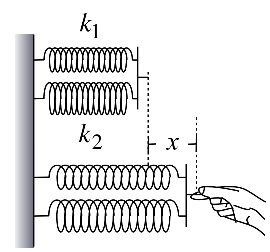
D) Write the expression for $\sum F_x$, and show that the plank undergoes simple harmonic motion horizontally.

E) Find the period of the simple harmonic motion.

Springs



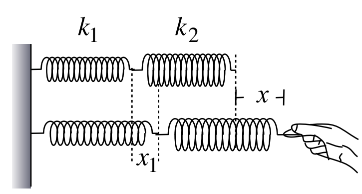
The most common example of SHM is a mass connected to a spring. We haven't mentioned springs so far, to emphasize the fact that SHM doesn't *necessarily* involve springs. An *ideal spring* is one that is considered *massless*, and obeys Hooke's Law. There is space between the coils of the spring, so it can be stretched or compressed from its equilibrium position. In this context the Hooke's Law constant k is often called the *spring constant* or *stiffness constant*. Consider the ideal spring to the right with spring constant k, fixed to a wall, and connected to a block of mass m on a frictionless surface. The drawings below it show the block held in place to the right and left of equilibrium by a distance A. When the block is released from rest at either position, it moves horizontally in SHM with amplitude A and period $T = 2\pi\sqrt{\frac{m}{k}}$.



Consider two ideal springs with constants k_1 and k_2 as shown. The springs are connected side by side, fastened to a wall, and stretched a distance x.

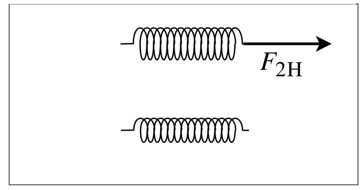
A) What is the force required to stretch the combined springs a distance x?

B) What is effective spring constant for the combined springs? Explain.



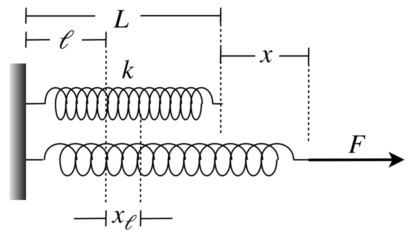
The springs are now connected end to end as shown, spring 1 is fastened to a wall, and the combined spring is stretched a distance x. Spring 1 stretches a distance x_1 as shown, and spring 2 stretches a distance x_2 (not shown).

A) In the space below, draw vectors showing the horizontal forces on the two springs. Draw these vectors *to scale*. The force on spring 2 by the hand is shown. Explain how you know the correct scale for the other forces.



B) What is effective spring constant for the combined springs?

When a spring of length L and stiffness k is cut to a fraction of its original length, it is possible to calculate the new stiffness constant of the smaller piece. The diagram to the right shows a force F that stretches the original spring by a distance x.

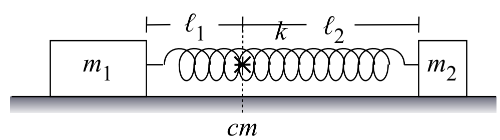


A) Consider a fraction of the original spring of length ℓ shown in the diagram. It is stretching a distance x_l. How does x_l compare to x?

B) What is the magnitude of the force that pulls the fraction of length ℓ to the right? Explain.

C) Express the stiffness constant k_ℓ of the fraction in terms of k, L and ℓ.

Two blocks of mass m_1 and m_2 are now fastened to the ends of the spring, as shown to the right. The distances ℓ_1 and ℓ_2 represent the distances of m_1 and m_2, respectively, from the center of mass cm of the system. When the blocks are pulled apart and released from rest, they undergo simple harmonic motion, but the center of mass remains stationary.



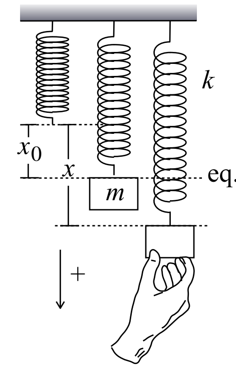
A) What is the stiffness constant k_1 of the portion of the spring of length ℓ_1?

B) Use the definition of center of mass to express k_1 in terms of k and the two masses.

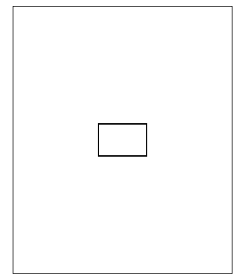
C) Since the center of mass of the system does not move, the block of mass m_1 oscillates as if it were connected to a spring of stiffness k_1, fastened to a wall. What is the period in terms of the two masses and k?

D) What is the mass of a single object that would oscillate with the same period from the original spring? (Do you recognize this expression?)

A block of mass m is hung from a spring of force constant k. It is lowered slowly until the spring stretches a distance x_0 and the block rests at equilibrium. In what follows, choose down as positive.



A) In the space below, draw and label the forces acting on the mass while at rest.



B) Express the value of x_0 in terms of m, g and k:

C) The mass is displaced from equilibrium and released. Write the equation that follows from Newton's Second Law:

D) Use the definition of acceleration to express the above as a second order differential equation in x (see p. 8.13):

The equation $x = A\cos(\omega t + \phi) + x_0$ describes a particle moving in SHM with an equilibrium position shifted from $x = 0$ to $x = x_0$, where $\omega = \sqrt{k/m}$ as before.

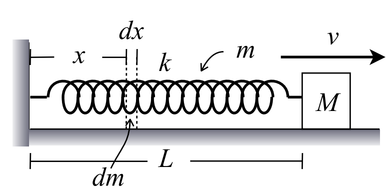
E) For a particle with this position function, express the velocity and the acceleration as functions of time:

F) Show that $x = A\cos(\omega t + \phi) + x_0$ is a solution to the second order differential equation in part D).

G) Choosing the equilibrium position ($x = x_0$) as the zero level of gravitational potential energy, write the expression relating the gravitational potential energy, the kinetic energy and the elastic potential energy:

H) Take the time derivative of every term, using the chain rule. Simplify and compare to the equation in part C).

Ideal springs are massless, but real springs have mass. Consider a block of mass M oscillating on the end of a spring of mass m. At a certain instant, the block has speed v as shown. We'll neglect the fact that a real spring would sag in the middle.



A) A small section of the spring at a distance x from the end has length dx and mass dm. In terms of v, what is the speed of this section?

B) What is the ratio of dm to m?

C) Write an expression for the differential kinetic energy dK of this section:

D) Integrate with appropriate limits to find the kinetic energy of the entire spring at this instant.

E) What is the mass of the block that would have the same kinetic energy as this system, but oscillating on a *massless* spring with the same spring constant?

Rotational (Angular) Simple Harmonic Motion

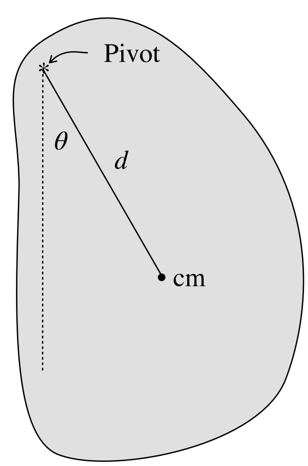
We have seen many parallels between linear and angular motion. SHM is no exception. Here is a summary of the definition of linear SHM, where we have written the amplitude A as $x_{\max}$:

> Linear SHM occurs when a restoring *force* is proportional to *displacement*; ie. $F = -kx$. This can be written as a second-order differential equation in x: $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$. The solution of this differential equation can be written: $x = x_{\max}\cos\left(\omega t + \phi\right)$, where $\omega = \sqrt{k/m}$.

Write the corresponding statement for rotational SHM (RSHM). Note that, in keeping with the tradition of using Greek letters for rotation, we use the letter κ (kappa) for the Hooke's Law constant.

Examples of Rotational SHM

1. The Physical Pendulum: When an object is suspended from a pivot point a distance d from its center of mass, it will be at equilibrium when the center of mass is directly under the pivot. If it is displaced slightly from equilibrium, the gravitational force supplies a restoring torque.



A) Write the expression for the restoring torque as a function of m, g, d and θ. Use the small angle approximation (see page 8.16) to show that the torque is proportional to θ.

B) Use the above expression to find κ, the proportionality constant, and express the period in terms of I, m, g and d:

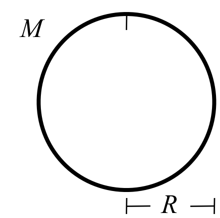
C) Consider the simple pendulum with small ball of mass m on the end of a string of length L as a special case of a physical pendulum. Show that your expression in part B) for the period yields the same result as on page 8.15.

2. A meter stick is suspended from one end as shown to the right. Recall that the rotational inertia of a stick about one end is $I = \frac{1}{3}ML^2$.

A) Find the period for small oscillations.

B) What is the length of the simple pendulum that has the same period as the meter stick?

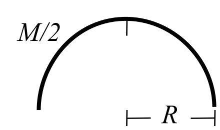
4. A hoop of mass M and radius R is suspended by a knife edge from a point on the circumference of the hoop.



A) Determine the rotational inertia of the hoop about the knife edge.

B) Determine the period for small oscillations.

The hoop is now cut in half. Recall that the center of mass of a semicircle is $\frac{2R}{\pi}$ from the center of the circle.



C) What is the rotational inertia of the semicircular hoop about its center?

D) Determine the rotational inertia of the semicircular hoop about the center of mass.

E) Determine the rotational inertia of the semicircular hoop about the knife edge.

F) Determine the period for small oscillations.

AP Physics C
Unit 8 Practice Test

Name ____________________

Part I
Multiple Choice

1. Two identical satellites X and Y are in circular orbits around the earth. The orbit of X has half the radius of the orbit of Y. Which of the following statements is/are correct?

   I The gravitational potential energy of X is less than that of Y.
   II The kinetic energy of X is less than that of Y.
   III The total energies of the two satellites are equal.

   A) I, II and III B) I and II only C) II and III only D) I only E) III only

2. The mass density of a certain planet has spherical symmetry, but varies in such a way that the mass inside any concentric spherical surface is proportional to the radius of the surface. The gravitational force F on a point mass inside the planet a distance r from its center is related to r as:

   A) $F \propto r^2$
   B) $F \propto r$
   C) $F \propto 1/r$
   D) $F \propto 1/r^2$
   E) F is independent of r

3. Two particles, each of mass m, are a distance d apart. To bring a third particle, also of mass m, from far away to the point midway between the two particles, an external agent does work given by:

   A) $4Gm^2/d$
   B) $-4Gm^2/d$
   C) $4Gm^2/d^2$
   D) $-4Gm^2/d^2$
   E) none of these

4. An object is dropped from an altitude of one earth radius above the earth's surface. If M is the mass of the earth and R is its radius, and we neglect air resistance, the speed of the object just before it hits the earth is given by:

   A) $\sqrt{\dfrac{GM}{R}}$

   B) $\sqrt{\dfrac{GM}{2R}}$

   C) $\sqrt{\dfrac{2GM}{R}}$

   D) $\sqrt{\dfrac{GM}{R^2}}$

   E) $\sqrt{\dfrac{GM}{2R^2}}$

5. The average distance from Mars to the sun is 1.52 times the distance from earth to the sun. The number of earth-years required for Mars to orbit about the sun is

   A) 1.52
   B) 2.31
   C) 1.23
   D) 1.87
   E) 1.32

6. If air friction is neglected, a 1 kg particle has an escape speed of about 7 miles per second at the surface of the earth. If the mass of the projectile is doubled,the escape speed is

A) 14 miles/sec
B) $7\sqrt{2}$ miles/sec
C) 7 miles/sec
D) $7\sqrt{2}$ miles/sec
E) 3.5 miles/sec

7. Suitable units for the universal gravitational constant G are:

A) kg•m/s^2 B) m/s^2 C) N•s/m D) kg•m/s E) m^3/(kg•s^2)

8. Two satellites are launched at a distance R from a planet of negligible radius. Both satellites are launched in the tangential direction. The first satellite launches correctly at a speed v_0 and enters a circular orbit. The second satellite, however, is launched at a speed $\frac{1}{2}v_0$. What is the minimum distance between the second satellite and the planet over the course of its orbit?

A) $\frac{1}{\sqrt{2}}R$

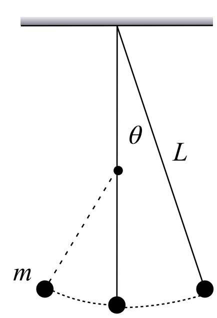
B) $\frac{1}{2}R$

C) $\frac{1}{3}R$

D) $\frac{1}{4}R$

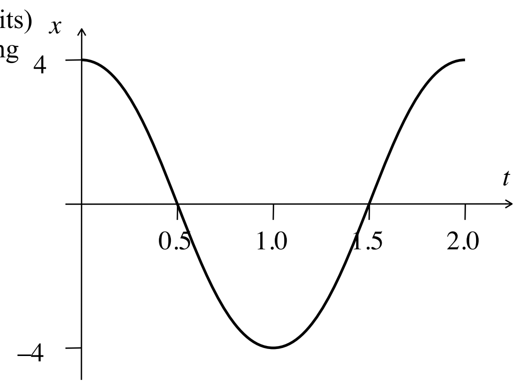
E) $\frac{1}{7}R$

9. A small mass m is suspended from a string of length L as shown to the right. It is swung through a small angle θ and released. Directly below the suspension point, and at the midpoint of the string, is a small peg, so that the pendulum swings as shown. Although the motion is not simple harmonic, it does have a period, which is closest to



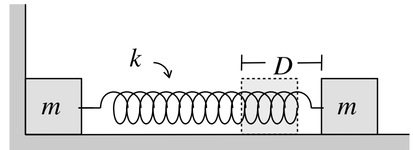
A) $1.57\sqrt{L/g}$
B) $3.14\sqrt{L/g}$
C) $4.71\sqrt{L/g}$
D) $5.36\sqrt{L/g}$
E) $6.28\sqrt{L/g}$

10. A particle moves in simple harmonic motion represented (in arbitrary units) by the graph of displacement vs. time to the right. Which of the following represents the *velocity* of the particle as a function of time?



A) $v(t) = 4\cos \pi t$
B) $v(t) = \pi \cos \pi t$
C) $v(t) = -\pi^2 \cos \pi t$
D) $v(t) = -4\sin \pi t$
E) $v(t) = -4\pi \sin \pi t$

For questions 11 - 14: A system consists of two blocks, each of mass m, connected by a spring of force constant k on a frictionless floor. The system is initially shoved against a wall so that the spring is compressed a distance D from its original uncompressed length. It is then released from this position with no initial velocity.



11. The maximum speed of the right-hand block, just before the left-hand block leaves the wall, is

A) $\sqrt{\dfrac{k}{m}}D$

B) $\sqrt{\dfrac{k}{2m}}D$

C) $\dfrac{1}{2}\sqrt{\dfrac{k}{m}}D$

D) $\dfrac{kD}{m}$

E) $\dfrac{kD}{2m}$

12. The speed of the center of mass of the system after the left-hand block is no longer in contact with the wall is

A) $\sqrt{\dfrac{k}{m}}D$ B) $\sqrt{\dfrac{k}{2m}}D$ C) $\dfrac{1}{2}\sqrt{\dfrac{k}{m}}D$ D) $\dfrac{kD}{m}$ E) $\dfrac{kD}{2m}$

13. The period of oscillation for the system when the left-hand block is no longer in contact with the wall is

A) $2\pi\sqrt{\dfrac{m}{k}}$ B) $2\pi\sqrt{\dfrac{m}{2k}}$ C) $2\pi\sqrt{\dfrac{2m}{k}}$ D) $2\pi\sqrt{\dfrac{k}{m}}$ E) $2\pi\sqrt{\dfrac{k}{2m}}$

14. If, instead, the floor were not frictionless, the minimum coefficient of kinetic friction between the blocks and the floor that would prevent the right-hand block from returning to its original position after being moved a distance D is

A) $\dfrac{kx^2}{2mg}$ B) $\dfrac{kx}{mg}$ C) $\dfrac{kx}{2mg}$ D) $\dfrac{mg}{2kx}$ E) $\dfrac{k}{4mgx}$

15. Three point masses m are attached together by identical springs. When placed at rest on a horizontal surface the masses form a triangle with side length L. When the assembly is rotated about its center at angular velocity ω, the masses form a triangle with side length $2L$. What is the spring constant k of the springs?

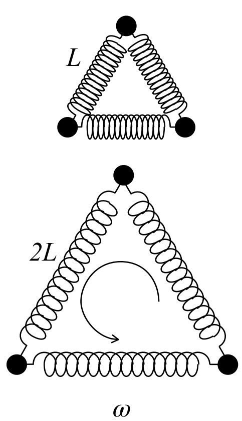
A) $2m\omega^2$

B) $\dfrac{2}{\sqrt{3}}m\omega^2$

C) $\dfrac{2}{3}m\omega^2$

D) $\dfrac{1}{\sqrt{3}}m\omega^2$

E) $\dfrac{1}{3}m\omega^2$



1.D 2.C 3.B 4.A 5.D 6.C 7.E 8.E 9.D 10.E 11.A 12.C 13.B 14.C 15.C

Part II

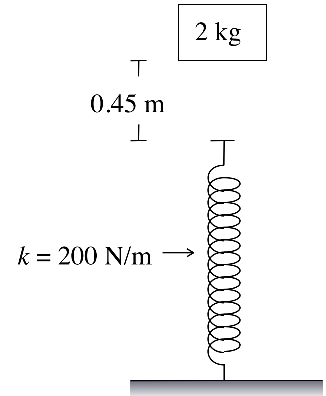
Show your work

Credit depends on the quality and clarity of your explanations

1. In March 1999 the Mars Global Surveyor (GS) entered its final orbit about Mars, sending data back to Earth. Assume a circular orbit with a period of 118 minutes and an orbital speed of 3.40×10^3 m/s. The mass of the GS is 930 kg and the radius of Mars is 3.43×10^6 m .

   A) Calculate the radius of the GS orbit.
   B) Calculate the mass of Mars.
   C) Calculate the total mechanical energy of the GS in this orbit.
   D) If the GS was to be placed in a lower circular orbit (closer to the surface of Mars), would the new orbital period of the GS be greater than or less than the given period? Justify your answer.
   E) In fact, the orbit the GS entered was slightly elliptical with its closest approach to Mars at 3.71×10^5 m above the surface and its furthest distance at 4.36×10^5 m above the surface. If the speed of the GS at closest approach is 3.40×10^3 m/s m/s , calculate the speed at the furthest point of the orbit.

2. A 2 kg block is dropped from a height of 0.45 m above an uncompressed spring, as shown to the right. The spring has a force constant of 200 N/m and negligible mass. The block strikes the end of the spring at time $t = 0$ and sticks to it.



A) Determine the speed of the block at the instant it hits the end of the spring.
B) Determine the period of the simple harmonic motion that ensues.
C) Determine the distance that the spring is compressed at the instant the speed of the block is maximum.
D) Determine the maximum compression of the spring.
E) Determine the amplitude of the simple harmonic motion.
F) Determine the phase constant of the simple harmonic motion.